Journal of Achievements in Materials and Manufacturing Engineering
VOLUME 18
ISSUE 1-2
September–October
2006

Nonlinear analysis of functionally graded beams

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Received 15.03.2006; accepted in revised form 30.04.2006

Analysis and modelling

ABSTRACT

Purpose: It is the intention of the present study to examine the effect of geometric nonlinearity on displacements and stresses in beams made of functionally graded materials (FGMs) subjected to thermo-mechanical loadings.
Design/methodology/approach: The nonlinear strain-displacement relations are used to study the effect of geometric nonlinearity. Temperature distribution through the thickness of the beams in thermal loadings is obtained by solving the one-dimensional heat transfer equation. Then the equilibrium equations are obtained within the framework of the first-order shear deformatyion beam theory (FSDBT) and then solved exactly and also by using a perturbation technique. The results obtained from these two methods are compared for various loadings and boundary conditions.
Findings: The numerical results showed that the nonlinearity effect on the displacements and stresses of the beams is significant. Also the effects of material constant n and the boundary conditions on the nonlinear bending behavior of the beams are determined.
Research limitations/implications: The exact solution method of nonlinear equilibrium equations can only be developed for composite beams with the same boundary conditions at the ends.
Practical implications: It is showed that for the maximum deflections greater than 0.3h a nonlinear solution is required.
Originality/value: The paper introduces a new method to obtain analytical solution for nonlinear equilibrium equations. This method can be used in developing higher-order shear deformation and layerwise theories.
Keywords: Applied mechanics; Functionally graded beams; Nonlinear analysis; Analytical solution

1. Introduction

The fast progress of modern high technology requires more and more new materials with various special properties or functions [1,2]. Under some severe environment such as super-high temperature, conventional materials may not service. A new material concept functionally graded materials (FGMs) has been proposed to meet the need [3] which usually comprises different material constituents such as ceramics and metal. FGMs have received considerable attention in many engineering applications since they were first reported in 1980s. FGMs are composite materials, microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. Compared with classical laminated composite materials, FGMs provide superior thermo-mechanical performances under given loading circumstances [4]. FGMs can be used to improve creep behavior [5], fracture toughness of machine tools [6], wear resistance, oxidation resistance of high temperature aerospace and automotive components and so on.

There are many studies concerning numerical solutions of thermo-mechanical responses of functionally graded (FG) beams and plates that account for moderately large rotations in the von-Kármán sense (see, e.g., [7-11]). But no research has been devoted so far for developing exact solution of nonlinear equilibrium equations of (FG) beams. To this end, a first-order shear deformation beam theory (FSDBT) is used to analyze displacements and stresses in beams made of functionally graded materials. The nonlinear strain-displacement relations are used to study the effect of geometric nonlinearity on displacements and stresses of the beams. The equilibrium equations are solved exactly and also by using a perturbation technique. The results obtained from these two methods are compared for various loading and boundary conditions. Finally, effects of considering geometric nonlinearity on deflections and stresses are determined.

2. Mathematical formulations

Consider a FG beam with length *L*, total thickness *h*, and width *b*. The bottom surface of the beam ($z=-h/2$) is subjected to a normal transverse load. It is assumed that a generic material property at a point *z* in FGMs is approximated by a power-law distribution in terms of the volume fractions of the constituents [8]. In the present study, the beam may be also subjected to a thermal loading. The variation of temperature is assumed to occur in the thickness direction only. The thermal analysis is carried out by first solving a simple steady state heat transfer equation through the thickness of the beam where the thermal boundary conditions are $T=T_c$ at $z=h/2$ and $T=T_m$ at $z=-h/2$.

Here a first-order shear deformation plate theory is used to derive first-order shear deformation beam theory. The displacement field is assumed as:

$$u(x,y,z)=u_0(x,y)+z\psi_x(x,y)$$
$$v(x,y,z)=v_0(x,y)+z\psi_x(x,y)\ ,\ w(x,y,z)=w(x,y) \tag{1}$$

where u_0, v_0, and w denote the displacements of a point on the middle plane of the plate. Also ψ_x and ψ_y are unknown functions which denote rotations of a cross-section about *y* and *x* axes, respectively. In the present study we wish to investigate the effect of geometric nonlinearity on the response quantities. Therefore, the von Kármán-type of geometric nonlinearity is taken into consideration in the strain-displacement relations. Substituting Equations (1) into the appropriate strain-displacement relations results in:

$$\varepsilon_x=\varepsilon_x^0+z\kappa_x,\quad \varepsilon_y=\varepsilon_y^0+z\kappa_y,\quad \varepsilon_z=0$$
$$\gamma_{yz}=\gamma_{yz}^0,\quad \gamma_{xz}=\gamma_{xz}^0,\quad \gamma_{xy}=\gamma_{xy}^0+z\kappa_{xy} \tag{2}$$

where

$$\varepsilon_x^0=u_{0,x}+(w_{,x})^2/2,\quad \kappa_x=\psi_{x,x}$$
$$\varepsilon_y^0=v_{0,y}+(w_{,y})^2/2,\quad \kappa_y=\psi_{y,y}$$
$$\gamma_{yz}^0=\psi_y+w_{,y},\quad \gamma_{xz}^0=\psi_x+w_{,x} \tag{3}$$
$$\gamma_{xy}^0=u_{0,y}+v_{0,x}+w_{,x}w_{,y},\quad \kappa_{xy}=\psi_{x,y}+\psi_{y,x}$$

Using the principle of minimum total potential energy, the equilibrium equations can be shown to be:

$$N_{x,x}+N_{xy,y}=0,\quad N_{xy,x}+N_{y,y}=0$$
$$M_{x,x}+M_{xy,y}-Q_x=0,\quad M_{xy,x}+M_{y,y}-Q_y=0 \tag{4}$$
$$Q_{x,x}+Q_{y,y}+\mathrm{N}(w)+q(x,y)=0$$

where

$$\mathrm{N}(w)=\left(N_x w_{,x}+N_{xy}w_{,y}\right)_{,x}+\left(N_{xy}w_{,x}+N_y w_{,y}\right)_{,y} \tag{5}$$

and $q(x,y)$ is the transverse load that is applied on the bottom surface of the plate. Also the force and moment resultants are defined as:

$$\left(N_x,N_y,N_{xy}\right)=\int_{-h/2}^{h/2}(\sigma_x,\sigma_y,\sigma_{xy})dz$$
$$\left(M_x,M_y,M_{xy}\right)=\int_{-h/2}^{h/2}(\sigma_x,\sigma_y,\sigma_{xy})zdz \tag{6}$$
$$\left(Q_x,Q_y\right)=\int_{-h/2}^{h/2}(\sigma_{xz},\sigma_{yz})dz$$

The linear constitutive relations are given by:

$$\sigma_x=Q_{11}(\varepsilon_x-\alpha\Delta T)+Q_{12}(\varepsilon_y-\alpha\Delta T)$$
$$\sigma_x=Q_{12}(\varepsilon_x-\alpha\Delta T)+Q_{22}(\varepsilon_y-\alpha\Delta T) \tag{7}$$
$$\sigma_{yz}=Q_{44}\gamma_{yz},\quad \sigma_{xz}=Q_{55}\gamma_{xz},\quad \sigma_{xy}=Q_{66}\gamma_{xy}$$

Upon substitution of Equations (7) into Equations (6), the force and moment resultants will be obtained which can be presented as follows:

$$N_x=A_{11}\varepsilon_x^0+A_{12}\varepsilon_y^0+B_{11}\kappa_x+B_{12}\kappa_y-N_x^T$$
$$N_y=A_{12}\varepsilon_x^0+A_{22}\varepsilon_y^0+B_{12}\kappa_x+B_{22}\kappa_y-N_y^T$$
$$M_x=B_{11}\varepsilon_x^0+B_{12}\varepsilon_y^0+D_{11}\kappa_x+D_{12}\kappa_y-M_x^T \tag{8a}$$
$$M_y=B_{12}\varepsilon_x^0+B_{22}\varepsilon_y^0+D_{12}\kappa_x+D_{22}\kappa_y-M_y^T$$
$$N_{xy}=A_{66}\gamma_{xy}^0+B_{66}\kappa_{xy},\quad M_{xy}=B_{66}\gamma_{xy}^0+D_{66}\kappa_{xy} \tag{8b}$$
$$Q_y=k^2A_{44}\gamma_{yz}^0,\quad Q_x=k^2A_{55}\gamma_{xz}^0 \tag{8c}$$

where

$$(A_{ij},B_{ij},D_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z,z^2)dz$$
$$(N_x^T,M_x^T)=\int_{-h/2}^{h/2}(Q_{11}+Q_{12})\alpha\Delta T(1,z)dz \tag{9}$$
$$(N_y^T,M_y^T)=\int_{-h/2}^{h/2}(Q_{12}+Q_{22})\alpha\Delta T(1,z)dz$$

and k^2(=5/6) is the shear correction factor. Next, in order to derive the beam theory it is assumed that $N_y=M_y=0$. By imposing this assumptions in Equations (8a) results in:

$$\begin{Bmatrix}N_x\\M_x\end{Bmatrix}=\begin{bmatrix}\bar{A}_{11}&\bar{B}_{11}\\\bar{B}_{11}&\bar{D}_{11}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^0\\k_x\end{Bmatrix}-\begin{Bmatrix}\bar{N}_x^T\\\bar{M}_x^T\end{Bmatrix} \tag{10}$$

where

$$\begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} - \begin{bmatrix} A_{12} & B_{12} \\ B_{12} & D_{12} \end{bmatrix} \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} \\ B_{12} & D_{12} \end{bmatrix} \tag{11}$$

$$\begin{Bmatrix} \bar{N}_x^T \\ \bar{M}_x^T \end{Bmatrix} = -\begin{bmatrix} A_{12} & B_{12} \\ B_{12} & D_{12} \end{bmatrix} \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix}^{-1} \begin{Bmatrix} N_y^T \\ M_y^T \end{Bmatrix} + \begin{Bmatrix} N_x^T \\ M_x^T \end{Bmatrix}$$

It is also assumed that all the force and moment resultants are functions of coordinate x only. Hence, Equations (4) are simplified as follows:

$$\frac{dN_x}{dx} = 0, \quad \frac{dN_{xy}}{dx} = 0, \quad \frac{dM_x}{dx} - Q_x = 0$$
$$\frac{dM_{xy}}{dx} - Q_y = 0, \quad \frac{dQ_x}{dx} + \frac{d}{dx}\left(N_x \frac{dw}{dx}\right) + q(x) = 0 \tag{12}$$

where $q(x)$ is the transverse load that is applied on the bottom surface of the beam.

3. Solution methodologies

In this section a beam subjected to a uniform transverse loads on its bottom surface and/or thermal loads is considered. In what follows two solution methodologies, exact and perturbation, are presented.

To obtain the exact solutions of Equations (12) the boundary conditions of the beam at $x = \pm L/2$ are assumed to be the same. By integrating the first equation of Equations (12) with respect to x results in:

$$\varepsilon_x^0 = \left(\bar{N}_x^T + N_x^0 - \bar{B}_{11} k_x\right) / \bar{A}_{11} \tag{13}$$

where N_x^0 is a constant of integration. With (13), Equations (3) and (8), with $\partial / \partial y = 0$, are substituted into Equations (12) to yield:

$$u_0' + (w')^2/2 = (\bar{N}_x^T + N_x^0 - \bar{B}_{11}\psi_x')/\bar{A}_{11} \tag{14a}$$
$$A_{66} v_0'' + B_{66} \psi_y'' = 0 \tag{14b}$$
$$\left(\bar{D}_{11} - \frac{\bar{B}_{11}^2}{\bar{A}_{11}}\right)\psi_x'' - k^2 A_{55}(\psi_x + w') = \frac{d\bar{M}_x^T}{dx} - \frac{\bar{B}_{11}}{\bar{A}_{11}} \cdot \frac{d\bar{N}_x^T}{dx} \tag{14c}$$
$$B_{66} v_0'' + D_{66} \psi_y'' - k^2 A_{44} \psi_y = 0 \tag{14d}$$
$$k^2 A_{55}(\psi_x' + w'') + N_x^0 w'' = -q(x) \tag{14e}$$

where a prime indicates an ordinary derivative with respect to x. Equations (14) are five linear ordinary differential equations with constant coefficients. It is noted that Equations (14b) and (14d) are both homogeneous and in terms of v_0 and ψ_y only. Since the corresponding boundary conditions are all homogeneous and in terms of v_0 and ψ_y only, the solution of Equations (14b) and (14d) is only a trivial one. The remaining equations can be solved analytically for any sets of boundary conditions in terms of the unknown constant N_x^0. After solving these equations we will use one more condition to find the final solutions. This condition is obtained by noting that, for example, for the C1-C1 and S1-S1 boundary types we have u_0=0 at $x = \pm L/2$ which will allow us to find N_x^0 in a trial and error process. Towards this end, we note that integrating Equation (14a) from 0 to $L/2$ results in:

$$u_0(L/2) = u_0(0) - \int_0^{L/2} \left[\bar{B}_{11}\psi_x'/\bar{A}_{11} + (w')^2/2\right] dx + N_x^0 L/2\bar{A}_{11} \tag{15}$$

Clearly, because of symmetry we have $u_0(0)=0$. Therefore, we conclude that:

$$\int_0^{L/2} \left[\left(\bar{N}_x^T - \bar{B}_{11}\psi_x'\right)/\bar{A}_{11} - (w')^2/2\right] dx = -N_x^0 L/2\bar{A}_{11} \tag{16}$$

Finally, by making the solutions of the Equations (14a), (14c) and (14e) to satisfy (16) in a trial and error process, we will obtain the exact value of N_x^0.

Next, the perturbation technique, Lindstedt-Poincaré method, is used to solve the three coupled nonlinear ordinary differential equations. Now we define w_0 as $w(0) = w_0$. Also the unknown variables are represented by the following expansions:

$$u_0(x) = u_1(x) w_0 + u_2(x) w_0^2 + u_3(x) w_0^3 + \cdots \tag{17a}$$
$$\psi_x(x) = \psi_{x1}(x) w_0 + \psi_{x2}(x) w_0^2 + \psi_{x3}(x) w_0^3 + \cdots \tag{17b}$$
$$w(x) = w_1(x) w_0 + w_2(x) w_0^2 + w_3(x) w_0^3 + \cdots \tag{17c}$$
$$C_i = C_{i1} w_0 + C_{i2} w_0^2 + C_{i3} w_0^3 + \cdots, \quad i = 1, 2, \cdots, 6 \tag{17d}$$

where w_0 is an unknown parameter which will be found at the end of analysis. Next, in mechanical loading ($\Delta T = 0$) we let:

$$q(x) = q_0 = q_1 w_0 + q_2 w_0^2 + q_3 w_0^3 + \cdots \tag{18}$$

And in thermal loading ($q(x)$=0) we consider the temperature of the top surface of the beam as ΔT. Finally, in this case, we let:

$$\Delta T = \Delta T_0 = \Delta T_1 w_0 + \Delta T_2 w_0^2 + \Delta T_3 w_0^3 + \cdots \tag{19}$$

where $\Delta T_0 = 300^0 C$ and q_i's and ΔT_i's are some unknown constants which will be found by imposing certain conditions. These conditions are found by noting that from $w(0) = w_0$ and (17c) we must conclude that:

$$w_1(0) = 1, \quad w_i(0) = 0, \quad i = 2, 3, \ldots \tag{20}$$

Substituting Equations (17) and (18) (or (19)) into equilibrium equations results in an infinite sets of coupled ordinary linear differential equations whose solutions can readily be obtained. Then constants q_i's (or ΔT_i's) are found by imposing the conditions in (20). Finally w_0 is found by numerically solving the polynomial equation in (18) (or (19)).

4. Results and discussion

Here we present results for a representative clamped-clamped beam (C1-C1) which its bottom surface is rich of Aluminum and the top surface is rich of Zirconia. For brevity, it is assumed that the beam is subjected to a uniform transverse load only and L/h=15 in all numerical examples. The mechanical properties of the constituents are E_m=70GPa, $\nu_m = 0.3$, E_c=151GPa, $\nu_c = 0.3$ where m and c indicate metal (i.e., Aluminum) and ceramic (i.e., Zirconia), respectively. In the numerical results the various non-dimensionalized parameters used are: length, $\bar{x} = x/L$, deflection, $\bar{w} = w/h$, longitudinal stress, $\bar{\sigma}_x = (\sigma_x h / q_0 L)$, load parameter, $\bar{q} = (q_0 L^4 / E_m h^4)$ where q_0 denotes the intensity of the applied uniform transverse load. For brevity all the numerical examples presented in what follows are for a FG beam with the power-law index n=3 subjected to the load parameter $\bar{q} = 36.16$. Figure 1 presents the variation of the center deflection of the beam versus the load parameter $\bar{q}$. It is seen that for the maximum deflections greater than 0.3h a nonlinear solution is required.

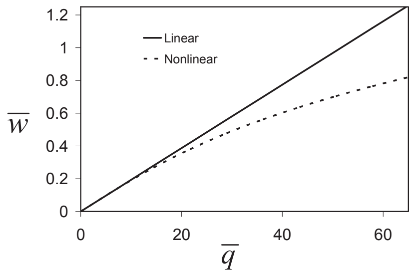

Fig. 1. Variation of center deflection of the FG beam versus the load parameter $\bar{q}$

Also variation of $\bar{\sigma}_x$ along the bottom surface of the beam is shown in Figure 2. It is observed from Figure 2 that there is excellent agreement between the results obtained form the exact method and those obtained from the perturbation technique. It is also seen that both the maximum deflection and normal stress in nonlinear analysis are smaller in magnitude in compared with linear analysis.

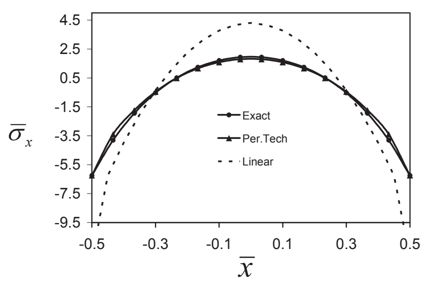

Fig. 2. Variation of $\bar{\sigma}_x$ along the bottom surface of the FG beam

5. Conclusions

In this study a first-order shear deformation beam theory is used to analyze displacements and stresses in functionally graded beams subjected to transverse loads on its bottom surfaces and/or thermal loads. The nonlinear strain-displacement relations are used to study the effect of the geometric nonlinearity. The material properties are assumed to vary according to a power-law distribution in terms of the volume fractions of the constituents. Temperature distributions through the thickness of the beams in thermal loading are obtained by solving the one-dimensional heat transfer equation. Next, the equilibrium equations are solved exactly and then by using a perturbation technique to verify the exact results. The exact solution method can be used for beams with the same boundary conditions, but the perturbation technique can be used for beams for the arbitrary boundary conditions. The numerical results obtained in the present study show that the nonlinearity effect on the displacements and stresses of the beams is significant. Also the effects of material constant n and the boundary conditions on the nonlinear bending behavior of the beams are determined.

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