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Dynamic response with arbitrary initial conditions using the FFT

U. Lee*, J. Cho
Department of Mechanical Engineering, Inha University, Incheon 402-751, South Korea
* Corresponding author: E-mail address: ulee@inha.ac.kr

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Analysis and modelling

ABSTRACT

Purpose: An FFT-based dynamic analysis method is proposed for damped linear discrete dynamic systems subjected to arbitrary nonzero initial conditions.
Design/methodology/approach: The DFT theory is used to develop an FFT-based spectral analysis method. The total dynamic response is considered as the sum of the forced vibration response part and the free vibration response part. The forced vibration response part is obtained from the dynamic stiffness matrix and the Fourier components of excitation force based on the concept of Duhamel's integral, and the free vibration response part is obtained by determining its integral constant to satisfy arbitrary initial conditions in the frequency-domain.
Findings: Through some numeral examples, the proposed FFT-based dynamic analysis method is shown to provide very successful solutions which satisfy all arbitrary non-zero initial conditions.
Research limitations/implications: (not applicable).
Practical implications: (not applicable).
Originality/value: The present FFT-based method is unique because it does not use the superposition of corrective free vibration solution or the pseudo-force concept used by other researchers to take into account the non-zero initial conditions.
Keywords: Numerical techniques; Spectral analysis method; Linear discrete system; FFT

1. Introduction

Due to the impressive progress in computer technologies during the last decades, there have been developed diverse computer-based numerical methods to obtain satisfactory solutions of differential equations. In the FFT-based spectral analysis method (SAM), the dependent variables of a set of ordinary differential equations are all transformed into the frequency-domain by using the discrete Fourier transforms (DFT) to transform the ordinary differential equations into a set of algebraic equations with frequency as the parameter. The algebraic equations are then solved for the Fourier (or spectral) components of dependent variables at each discrete frequency. As the final step, the time-domain responses are reconstructed from the Fourier components by using the inverse discrete Fourier transforms (IDFT). In practice, the FFT is used to carry out the DFT or IDFT. As the FFT is a remarkably efficient computer algorithm, it can offer an enormous reduction in computer time and also can increase solution accuracy [1].

The FFT-based SAM has been well applied to the prediction of the steady-state responses of dynamic systems [2, 3]. However the application of the FFT-based SAM to the transient responses has been limited to the dynamic systems with null initial conditions. As an effort to deal with dynamic systems with nonzero initial conditions, Veletsos and Ventura [4] introduced a DFT-based approach to calculate the transient responses of a linear 1-DOF system. Their procedure involves the superposition of a corrective free vibration solution which effectively transforms the steady-state response to the desired transient response. Later Mansur *et al*. [4] used the pseudo-force concept to take into account non-zero initial conditions in the DFT-based frequency-domain analysis of an FEM model.

The purpose of this paper is to present an FFT-based SAM for damped linear discrete dynamic systems subjected to arbitrary nonzero initial conditions. The present FFT-based SAM is unique because it does not use the superposition of corrective, free vibration solution to match the initial conditions as in references [4], or the pseudo-force concept to take into account the non-zero initial conditions [5].

2. Dynamic response

The vibration of a damped m-DOFs dynamic system can be represented by

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t)+\boldsymbol{C}\dot{\boldsymbol{u}}(t)+\boldsymbol{K}\boldsymbol{u}=\boldsymbol{f}(t) \tag{1}$$

with initial conditions

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \text{ and } \dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}_0 \tag{2}$$

Mathematically the dynamic response of a system can be expressed as the sum of the forced vibration response part purely determined by the excitation force and the free vibration response part purely determined by non-zero initial conditions:

$$\boldsymbol{u}(t)=\boldsymbol{u}_f(t)+\boldsymbol{u}_h(t) \tag{3}$$

where $\boldsymbol{u}(t)$ is the total dynamic response, $\boldsymbol{u}_p(t)$ the forced vibration response part satifying null initial conditions, and $\boldsymbol{u}_h(t)$ is the free vibration response part to be determined to satisfy arbitrary nonzero initial conditions.

3. Forced vibration response part

Duhamel integral provides the forced vibration response of a system subjected to null initial conditions [1, 2]. The Fourier transforms of the Duhamel integral simply says that the Fourier transforms of the forced vibration response is the simple product of the Fourier transforms of the unit impulse response function (*i.e.*, frequency response function) and the Fourier transforms of excitation force. Assume that the force vector $\boldsymbol{f}(t)$ and the forced vibration response part $\boldsymbol{u}_f(t)$ can be represented in the spectral forms as

$$\boldsymbol{f}(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\boldsymbol{F}_n e^{i\omega_n t_r}, \quad \boldsymbol{u}_f(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\boldsymbol{P}_n e^{i\omega_n t_r} \tag{4}$$

where r = 1, 2, …, N-1. Applying Eq. (4) into Eq. (1) gives

$$\boldsymbol{P}_n=\boldsymbol{D}(\omega_n)^{-1}\boldsymbol{F}_n, \quad \boldsymbol{P}_{N-n}=\boldsymbol{P}_n^*, \quad (n = 1, 2, \ldots, \mathrm{N}/2) \tag{5}$$

where $\boldsymbol{D}(\omega)$ is the dynamic stiffness matrix defined by

$$\boldsymbol{D}(\omega)=\boldsymbol{K}+i\omega\boldsymbol{C}-\omega^2\boldsymbol{M} \tag{6}$$

Notice that the symbol (*) used throughout this paper denotes the complex conjuate of a complex quantity. Once the Fourier components $\boldsymbol{P}_n$ are computed from Eq. (5), the forced vibration response part can be computed by using the IFFT as follows:

$$\boldsymbol{u}_f(t)\Leftarrow IFFT\{\boldsymbol{P}_n\} \tag{7}$$

4. Free vibration response part

The free vibration response $\boldsymbol{u}_h(t)$ satisfies the homogeneous matrix equation of motion, which can be reduced from Eq. (1) as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_h(t)+\boldsymbol{C}\dot{\boldsymbol{u}}_h(t)+\boldsymbol{K}\boldsymbol{u}_h=\boldsymbol{0} \tag{8}$$

4.1. Proportional damping

If $\boldsymbol{C}$ is the proportional damping matrix, Eq. (8) can be decoupled by using the modal matrix $\boldsymbol{\Phi}$ into the form as

$$\ddot{x}_{hk}+2\xi_k\Omega_k\dot{x}_{hk}+\Omega_k^2 x_{hk}=0, \quad (k = 1, 2, \ldots, m) \tag{9}$$

where Ω_{lk} and ξ_k are the kth natural frequency and modal damping ratio, respectively. Equation (9) can be solved for

$$x_{hk}(t)=a_k e^{\lambda_{k1}t}+a_k^* e^{\lambda_{k1}^* t}, \quad (k = 1, 2, \ldots, m) \tag{10}$$

where λ_{k1} and λ_{k2} are the roots of the characteritic equation for Eq. (9) and a_k are constants tot be determined by initial conditions. Accordingly the solution of Eq. (8) can be expressed as

$$\boldsymbol{u}_h(t_r)=\frac{1}{N}\boldsymbol{\Phi}\sum_{n=0}^{N-1}\boldsymbol{H}_n e^{i\omega_n t_r} \quad (r = 0, 1, 2, \ldots, N\text{-}1) \tag{11}$$

where

$$\begin{aligned}
&\boldsymbol{H}_n=\boldsymbol{X}_n\boldsymbol{a}+\boldsymbol{Y}_n\boldsymbol{a}^*, \quad \boldsymbol{a}=\{a_1 \quad a_2 \quad \cdots \quad a_m\}^T\\
&\boldsymbol{X}_n=diag[X_{kn}], \quad \boldsymbol{Y}_{kn}=diag[Y_{kn}], \quad \boldsymbol{\Lambda}_{kn}=diag[\Lambda_{kn}]\\
&X_{kn}=\frac{1-e^{\alpha_{kn}N}}{1-e^{\alpha_{kn}}}, \quad Y_{kn}=\frac{1-e^{\beta_{kn}N}}{1-e^{\beta_{kn}}}\\
&\alpha_{kn}=(\lambda_{k1}-i\omega_n)\Delta, \quad \beta_{kn}=(\lambda_{k1}^*-i\omega_n)\Delta
\end{aligned} \tag{12}$$

Substitute the total dynamic response Eq. (3) determined by Eqs. (7) and (11) into the initial conditions Eq. (2) and apply Eq. (12) to solve for $\boldsymbol{a}$ as

$$\boldsymbol{a}=\frac{i}{2}\boldsymbol{R}\left(\boldsymbol{\Lambda}_1^*\boldsymbol{d}-\boldsymbol{v}\right) \tag{13}$$

where

$$\boldsymbol{d}=N\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u}_0, \quad \boldsymbol{v}=N\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{u}}_0, \quad \boldsymbol{R}=diag[R_k] \tag{14}$$

$$R_k = \frac{1}{Imaginary(\lambda_{k1})\overline{X}_k}, \quad \overline{X}_k = \sum_{n=0}^{N-1} X_{kn}, \quad (k=1,2,\ldots,m)$$

Once $\boldsymbol{a}$ is computed from Eq. (19), $\boldsymbol{H}_n$ are computed first from Eq. (12) and then use the IFFT to compute

$$\boldsymbol{u}_h(t) \Leftarrow \quad \Phi\, IFFT\{\boldsymbol{H}_n\} \tag{15}$$

4.2. Non-proportional damping

If $\boldsymbol{C}$ is the non-proportional damping matrix, Eq. (8) cannot be decoupled by using the modal decomposition analysis. Thus assume the solution of Eq. (8) as

$$\boldsymbol{u}_h(t) = \boldsymbol{A}\, e^{\lambda t} \text{ or } u_{hk}(t) = a_k\, e^{\lambda t}, \quad (k=1,2,\ldots,m) \tag{16}$$

Substituting Eq. (16) into Eq. (8) yield an eigenvalue problem. For the existence of non-trivial solution of the eigenvalue problem, one can derive a 2m-degree algebraic equation for eigenvalue λ. The eigenvlues will appear in the complex conjugate pairs for underdamped systems [6], because all coefficients of the algebraic equation are real. Thus, the eigenvalues can be written as

$$\lambda_j = \xi_j + i\Omega_j, \quad \lambda_j^* = \xi_j - i\Omega_j, \quad (j=1,2,\ldots,m) \tag{17}$$

where Ω_j represents the natural frequency and ξ_j the rate of exponential decay of the jth vibration mode. The ratio between the components a_k of the jth eigenvector are given by [6]

$$\frac{a_1}{C_{j1}} = \frac{a_2}{C_{j2}} = \cdots = \frac{a_m}{C_{jm}} = z_j \tag{18}$$

where C_{jk} is the co-factor of the jth row of the determinant of the eigenvalue problem for a particular λ_j, and z_j is an arbitrary complex number. From Eq. (16), the kth component of the jth vibration mode corresponding to λ_j is given by

$$u_{hk}(t) = \sum_{j=1}^{2m} b_j\, u_{hk}^j = \sum_{j=1}^{m} B_j\, C_{jk}\, e^{\lambda_j t} + \sum_{j=1}^{m} B_j^* C_{jk}^*\, e^{\lambda_j^* t} \tag{19}$$

By using the DFT theory, the spectral forms of $u_{hk}(t)$ can be written as

$$\boldsymbol{u}_h(t_r) = \frac{1}{N} \sum_{r=0}^{N-1} \boldsymbol{H}_n\, e^{i\omega_n t_r} \tag{20}$$

where

$$\boldsymbol{H}_n = \{H_{1n} \quad H_{2n} \quad \cdots \quad H_{mn}\}^{\mathrm{T}}$$
$$H_{kn} = \sum_{j=1}^{m} \left(B_j\, X_{jkn} + B_j^*\, Y_{jkn}\right) \tag{21}$$

$$X_{jkn} = C_{jk} \frac{1-e^{(\lambda_j - i\omega_n)\Delta N}}{1-e^{(\lambda_j - i\omega_n)\Delta}}, \quad Y_{jkn} = C_{jk}^* \frac{1-e^{(\lambda_j^* - i\omega_n)\Delta N}}{1-e^{(\lambda_j^* - i\omega_n)\Delta}}$$

Substitute the dynamic response Eq. (3) determined by Eqs. (8) and (20) into the initial conditions Eq. (2) and apply Eq. (21) to obtain

$$\overline{\boldsymbol{X}}\boldsymbol{B} + \overline{\boldsymbol{Y}}\boldsymbol{B}^* = \boldsymbol{d}, \quad \widetilde{\boldsymbol{X}}\boldsymbol{B} + \widetilde{\boldsymbol{Y}}\boldsymbol{B} = \boldsymbol{v} \tag{22}$$

where

$$\overline{\boldsymbol{X}} = \sum_{n=0}^{N-1} \boldsymbol{X}_n, \quad \overline{\boldsymbol{Y}} = \sum_{n=0}^{N-1} \boldsymbol{Y}_n, \quad \widetilde{\boldsymbol{X}} = \overline{\boldsymbol{X}}\boldsymbol{\Lambda}, \quad \widetilde{\boldsymbol{Y}} = \overline{\boldsymbol{Y}}\boldsymbol{\Lambda}^* \tag{23}$$

$$\boldsymbol{X}_n = \lfloor X_{jkn} \rfloor, \quad \boldsymbol{Y}_n = \lfloor Y_{jkn} \rfloor, \quad \boldsymbol{\Lambda} = diag[\lambda_k]$$

The constants vectors $\boldsymbol{B}$ can be solved from Eq. (22) as

$$\boldsymbol{B} = \left(\widetilde{\boldsymbol{X}} - \widetilde{\boldsymbol{Y}}\,\overline{\boldsymbol{Y}}^{-1}\overline{\boldsymbol{X}}\right)^{-1}\left(\boldsymbol{v} - \widetilde{\boldsymbol{Y}}\,\overline{\boldsymbol{Y}}^{-1}\boldsymbol{d}\right) \tag{24}$$

Once the constants vector $\boldsymbol{B}$ is computed from Eq. (24), the Fourier components $\boldsymbol{H}_n$ are computed first from Eq. (21). Then one can compute the free vibration response part $\boldsymbol{u}_h(t)$ by using the IFFT algorithm as follows:

$$\boldsymbol{u}_h(t) \Leftarrow \quad IFFT\{\boldsymbol{H}_n\} \tag{25}$$

5. Numerical examples and discussion

To evaluate the present (FFT-based) SAM, two damped three-DOFs dynamic systems are considered as the example problems. The first one is the case with the proportional damping as

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 3c & -2c & 0 \\ -2c & 3c & -c \\ 0 & -c & 3c \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix} \tag{26}$$

The second one is the case with the non-proportional damping. The initial conditions for both example cases are given by

$$\{u_1, u_2, u_3\} = \{1, 0, 0\}\ (\mathrm{mm}), \quad \{\dot{u}_1, \dot{u}_2, \dot{u}_3\} = \{0, 1, 0\}\ (\mathrm{mm/s}) \tag{27}$$

Figure 1 compares the dynamic responses obtained by three different solution methods for the case of proportional damping. The dynamic responses exactly obtained by the modal analysis method are used as the reference solutions to evaluate the present SAM. The DFT period T = 4.8 seconds and the number of samples $N = 2^{11}$ are used for the present SAM, whereas the time increment Δt = 0.00234 seconds is used for Runge-Kutta method. The present SAM is found to provide accurate solutions which are very close to the exact reference solutions and also to the numerical solutions obtained by Runge-Kutta method. Figure 2 compares the dynamic responses obtained by the present SAM and the Runge-Kutta method for the dynamic system with non-proportional damping. One may find from Figure 2 that the present SAM certainly provides the dynamic responses which are very close to those obtained by Runge-Kutta method. Figure 3 shows the convergence of the dynamic response $u_1(t)$ as the number of samples N is increased. As expected, more accurate result can be obtained by increasing N for a fixed time window, T = 4.8 seconds. Finally, it is worthwhile to confirm from Figure 1 through Figure 3 that the present SAM certainly captures all non-zero initial conditions exactly in the dynamic responses, which is one of major motivations of the present paper.

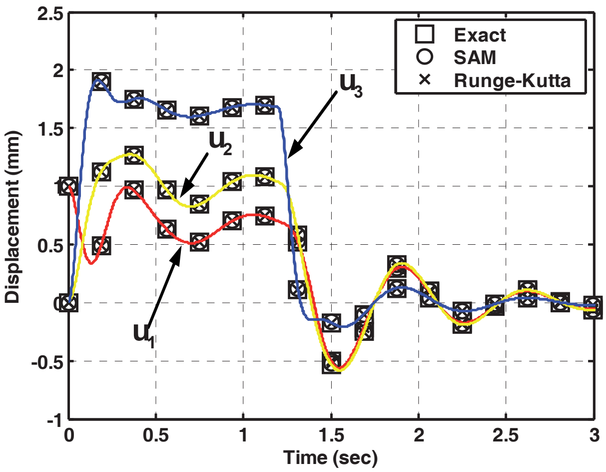

Fig. 1. Dynamic responses with proportional damping

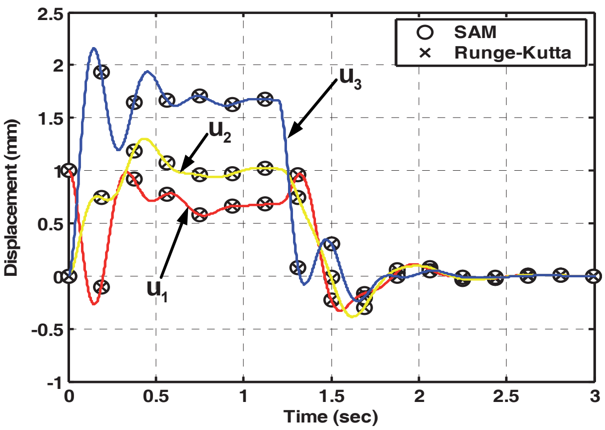

Fig. 2. Dynamic responses with non-proportional damping

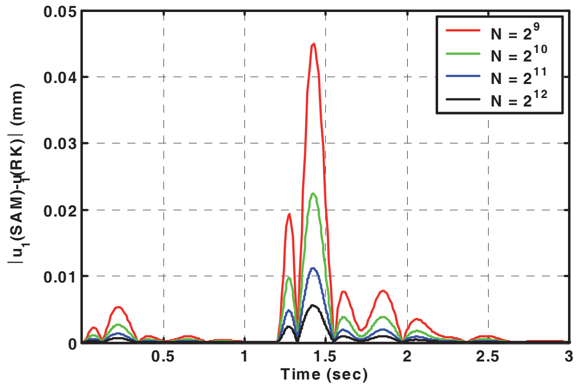

Fig. 3. Dynamic responses with non-proportional damping by the present SAM vs. the sampling number N

6. Conclusions

In this paper, an FFT-based SAM is developed to obtain the dynamic responses of a damped linear discrete dynamic system, subjected to arbitrary nonzero initial conditions. It is numerically shown that, by choosing a proper sampling number for a given time window, sufficiently accurate solutions can be obtained by using the present SAM, when compared with the solutions obtained by the modal analysis method and Runge-Kutta method.

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