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FFT-based spectral dynamic analysis for linear discrete dynamic systems

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Analysis and modelling

ABSTRACT

Purpose: An FFT-based spectral dynamic analysis method is developed for the viscously damped, linear discrete dynamic systems subjected to nonzero initial conditions.

Design/methodology/approach: The discrete Fourier transform (DFT) theory is used to develop a spectral dynamic analysis method. The dynamic response of a linear system is assumed as the sum of the forced and free vibration response parts. The forced vibration response part is obtained by convolving the dynamic stiffness matrix and Fourier components of excitation force through the Duhamel's integral, and the free vibration response part is obtained by determining its integral constants so as to satisfy initial conditions in frequency-domain.

Findings: It is shown through some numeral examples that the proposed FFT-based spectral dynamic analysis method provides the solutions which accurately satisfy all initial conditions.

Practical implications: This analysis method is applicable to viscously damped, linear discrete dynamic systems subjected to nonzero arbitrary initial conditions. In this study, two types of viscous damping are considered: proportional damping and non-proportional damping.

Originality/value: The FFT-based spectral dynamic analysis method proposed in this paper is unique because the pseudo-force concept or the superposition of corrective free vibration solution used by other researchers is not used to take into account non-zero initial conditions.

Keywords: Numerical techniques; Linear discrete system; Spectral analysis method; FFT

1. Introduction

During last three decades, diverse computer-based numerical methods [1-6] have been developed to obtain extremely accurate solutions for the large degrees of freedom (DOFs) systems due to remarkable progress in computer technologies. For example, there are two groups of numerical methods: time-domain methods and frequency-domain methods. The time-domain methods include the direct integration method, the modal analysis method, and the discrete-time system method [1]. The FFT-based spectral analysis method (SAM) is one of frequency-domain methods [2, 3].

In the FFT-based SAM, the dependent variables of a set of ordinary differential equations are all transformed into the frequency-domain by using DFT to transform the ordinary differential equations into a set of algebraic equations with the frequency as a parameter. The algebraic equations are then solved for the Fourier components of dependent variables at each discrete frequency. As the final step, the time-domain responses are reconstructed from the Fourier components by using the inverse discrete Fourier transforms (IDFT). In practice, the FFT is used to carry out the DFT or IDFT. As the FFT is a remarkably efficient computer algorithm, it can offer an enormous reduction in computer time and also can increase solution accuracy [7].

The FFT-based SAM has been known to be very useful especially in the following situations [2, 3]: (1) when it is more easier to derive the constitutive equation of a material in the frequency-domain rather than in the time-domain, (2) when the frequency-dependent spectral element or dynamic stiffness model is used as a structure model, (3) when the modern data acquisition systems are used to store digitized data through the analogue-to-digital converters, (4) when the excitation forces are so complicated that one has to use numerical integration to obtain the dynamic responses by using the excitation values at a discrete set of instants,. So it may be worthy to mention that the development of an FFT-based SAM is not necessary to compete with the exiting time-domain methods such as Runge-Kutta method, for instance. The FFT-based SAM will be especially useful in the aforementioned situations in which the frequency-domain analysis is more desirable.

The FFT-based SAM has been well applied to the prediction of the steady-state responses of dynamic systems [2, 8]. However the application of the FFT-based SAM to the transient responses has been limited to the dynamic systems with null initial conditions. As an effort to deal with dynamic systems with nonzero initial conditions, Veletsos and Ventura [9] introduced a DFT-based approach to calculate the transient responses of a linear 1-DOF system. Their procedure involves the superposition of a corrective free vibration solution which effectively transforms the steady-state response to the desired transient response. Later Mansur *et al*. [10] used the pseudo-force concept to take into account non-zero initial conditions in the DFT-based frequency-domain analysis of an FEM model.

The purpose of this paper is to present an FFT-based SAM for damped linear discrete dynamic systems subjected to arbitrary nonzero initial conditions by extending and further detailing the previous work [11]. The present FFT-based SAM is unique because it does not use the superposition of corrective, free vibration solution to match the initial conditions as in references, or the pseudo-force concept to take into account the non-zero initial conditions [9, 10]. To evaluate the FFT-based SAM proposed in this paper, the forced vibration of a viscously damped dynamic system is considered as an illustrative example.

2. The DFT theory

As we use the DFT as a key mathematical tool for developing a FFT-based SAM, we provide a brief review on the DFT theory in the following. Consider a periodic function of time $x(t)$ with period T. The periodic function $x(t)$ can be then expressed into a Fourier series as

$$x(t)=a_0+2\sum_{n=0}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right)=\sum_{n=-\infty}^{\infty}X_n e^{i\omega_n t} \tag{1}$$

where $i=\sqrt{-1}$, $\omega_n=n(2\pi/T)=n\omega_1$ are the discrete frequencies, and X_n are constant Fourier components given by

$$X_n=a_n-ib_n=\frac{1}{T}\int_0^T x(t)e^{-i\omega_n t}dt \quad (n=0,1,2,\ldots,\infty) \tag{2}$$

Equations (1) and (2) are the continuous Fourier transforms pair for a periodic function.

Although $x(t)$ is a continuous function of time, it is often the case that only sampled values of the function are available, in the form of a discrete time series $\{x(t_r)\}$. If N is the number of samples, all equally spaced with a time interval equal to $\Delta = T/N$, the discrete time series are given by $x_r = x(t_r)$, where $t_r = r\Delta$ and $r = 0, 1, 2, \ldots, N-1$. The integral in Eq. (2) may be replaced approximately by the summation

$$X_n=\sum_{r=0}^{N-1}x(t_r)e^{-i\omega_n t_r} \quad (n=0,1,2,\ldots,N-1) \tag{3}$$

which is the discrete Fourier transforms (DFT) of the discrete time series $\{x_r\}$. Any typical value x_r of the series $\{x_r\}$ can be given by the inverse formula

$$x(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}X_n e^{i\omega_n t_r} \quad (r=0,1,2,\ldots,N-1) \tag{4}$$

which is the inverse discrete Fourier transforms (IDFT). Thus, Eq. (3) and Eq. (4) represent the DFT-IFFT fair. Even though Eq. (3) is an approximation of Eq. (2), it is important to note that it allows all discrete time series $\{x_r\}$ to be regained *exactly* [7]. The Fourier components X_n in Eq. (4) are arranged as $X_{N-n}=X_n^*$ ($n = 0, 1, 2, \ldots, N/2$), where X_n^* represents the complex conjugate of X_n. Note that $X_{N/2}$ corresponds to the highest frequency $\omega_{N/2} = (N/2)\omega_1$, which is called the Nyquist frequency.

The fast Fourier transforms (FFT) is an ingenious highly efficient computer algorithm developed to perform the numerical operations required for a DFT or IDFT, reducing the computing time drastically by the order of $N/\log_2 N$. It should be pointed out that while the FFT-based spectral analysis uses a computer, it is not a numerical method in the usual sense, because the analytical descriptions of Eqs. (3) and (4) are still retained. Further details of DFT and FFT can be found in reference [2, 7].

3. Dynamic response

The equations of motion for a damped *m*-DOFs system can be expressed in the general form as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t)+\boldsymbol{C}\dot{\boldsymbol{u}}(t)+\boldsymbol{K}\boldsymbol{u}=\boldsymbol{f}(t) \tag{5}$$

with initial conditions

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \text{ and } \dot{\boldsymbol{u}}(0)=\dot{\boldsymbol{u}}_0 \tag{6}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass matrix, damping matrix, and stiffness matrix, respectively, $\boldsymbol{u}(t)$ is the DOFs vector, and $\boldsymbol{f}(t)$ is the excitation force vector.

Mathematically the solution of a differential equation can be obtained by summing its particular solution and homogeneous (complementary) solution. Since the integral constants of the homogeneous solution are determined to make the soltuion satisfy all initial conditions, the homogeneous solution consists of two parts: one of which depends only on the external force and the other one only on the non-zero initial conditions. So the dynamic response of a system can be re-expressed as the sum of the forced vibration response part determined purely by the excitation force to satisfy null initial conditions and the free vibration response part determined purely by only non-zero initial conditions:

$$\boldsymbol{u}(t)=\boldsymbol{u}_f(t)+\boldsymbol{u}_h(t) \tag{7}$$

where $\boldsymbol{u}(t)$ is the total dynamic response, $\boldsymbol{u}_p(t)$ is the forced vibration response part satisfying null initial conditions, and $\boldsymbol{u}_h(t)$ is the free vibration response part which will be determined to satisfy arbitrary nonzero initial conditions. The representation Eq. (7) for the total dynamic response is different from that used in the previous works [14, 15].

4. Forced vibration response part

Forced vibration response of a linear system subjected to null initial conditions can be readily obtained by using the Duhamel integral [2, 7]. The Fourier transforms of the Duhamel integral show that the Fourier transformed forced vibration response is the simple product of the Fourier transformed unit impulse response function and the Fourier transformed excitation force. Assume that the excitation force vector f(t) and the forced vibration response part $u_f(t)$ can be expressed in the spectral forms as

$$\boldsymbol{f}(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\boldsymbol{F}_n e^{i\omega_n t_r}$$
$$\boldsymbol{u}_f(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\boldsymbol{P}_n e^{i\omega_n t_r} \tag{8}$$

where r = 1, 2, …, N-1. Applying Eq. (8) into Eq. (5) gives

$$\boldsymbol{P}_n=\boldsymbol{D}(\omega_n)^{-1}\boldsymbol{F}_n, \quad \boldsymbol{P}_{N-n}=\boldsymbol{P}_n^* \qquad (n = 1, 2, \ldots, N/2) \tag{9}$$

where $\boldsymbol{D}(\omega)$ is the dynamic stiffness matrix defined by

$$\boldsymbol{D}(\omega)=\boldsymbol{K}+i\omega\boldsymbol{C}-\omega^2\boldsymbol{M} \tag{10}$$

Notice that the symbol (*) used throughout this paper denotes the complex conjuate of a complex quantity.

Once the Fourier components $\boldsymbol{P}_n$ are computed from Eq. (9) for a given force vector $\boldsymbol{f}(t)$, the forced vibration response part can be computed by using the IFFT as follows:

$$\boldsymbol{u}_f(t)\Leftarrow IFFT\{\boldsymbol{P}_n\} \tag{11}$$

From Eq. (8), the time derivative of $\boldsymbol{u}_f(t)$ can be obtained as

$$\dot{\boldsymbol{u}}_f(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\overline{\boldsymbol{P}}_n e^{i\omega_n t_r} \tag{12}$$

where

$$\overline{\boldsymbol{P}}_n=(i\omega_n)\boldsymbol{P}_n, \quad \overline{\boldsymbol{P}}_{N-n}=\overline{\boldsymbol{P}}_n^* \quad (n = 1, 2, \ldots, N/2) \tag{13}$$

5. Free vibration response part

As the free vibration response $\boldsymbol{u}_h(t)$ satisfies the homogeneous matrix equation of motion, it can be solved from

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_h(t)+\boldsymbol{C}\dot{\boldsymbol{u}}_h(t)+\boldsymbol{K}\boldsymbol{u}_h=\boldsymbol{0} \tag{14}$$

5.1. Proportional damping

When $\boldsymbol{C}$ is a proportional damping matrix, Eq. (14) can be decoupled by using the modal matrix. Thus, we assume the solution of Eq. (14) in the form

$$\boldsymbol{u}_h(t)=\boldsymbol{\Phi}\,\boldsymbol{x}_h(t) \tag{15}$$

where $\boldsymbol{\Phi}$ is the modal matrix which is the collection of normal modes satisfying the orthonormality properties

$$\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}=\boldsymbol{I}, \qquad \boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi}=\boldsymbol{\Omega}^2 \tag{16}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{\Omega}^2$ is the diagonal matrix defined by

$$\boldsymbol{\Omega}^2=\begin{bmatrix}\Omega_1^2 & & & \\ & \Omega_2^2 & & \\ & & \ddots & \\ & & & \Omega_m^2\end{bmatrix} \tag{17}$$

where $\Omega_1 \le \Omega_2 \le \ldots \le \Omega_m$ are the natural frequencies. By substituting Eq. (15) into Eq. (14) and by applying Eq. (16), one may obtain a set of decoupled modal equations as

$$\ddot{x}_{hk}+2\xi_k\Omega_k\dot{x}_{hk}+\Omega_k^2 x_{hk}=0 \qquad (k = 1, 2, \ldots, m) \tag{18}$$

where ξ_k is the kth modal damping ratio.

Substituting assumed solution $x_{hk}(t) = \exp(\lambda_k t)$ into Eq. (18) yields a characteristic equation:

$$\Delta(\lambda)=\lambda_k^2+2\xi_k\Omega_k\lambda_k+\Omega_k^2=(\lambda_k-\lambda_{k1})(\lambda_k-\lambda_{k2})=0 \tag{19}$$

Where λ_{k1} and λ_{k2} are the characteristic values and they are complex conjugates of each other. The free vibration response part $x_{hk}(t)$ can be then obtained in the form

$$x_{hk}(t)=a_k e^{\lambda_{k1}t}+a_k^* e^{\lambda_{k1}^* t} \qquad (k = 1, 2, \ldots, m) \tag{20}$$

where the constants a_k and a_k^* must be determined to satisfy initial conditions given by Eq. (6). The time derivative of Eq. (20) can be readily obtained as

$$\dot{x}_{hk}(t)=a_k\lambda_{k1}e^{\lambda_{k1}t}+a_k^*\lambda_{k1}^* e^{\lambda_{k1}^* t} \qquad (k = 1, 2, \ldots, m) \tag{21}$$

Represent $x_{hk}(t)$ and $\dot{x}_{hk}(t)$ into the spectral forms as

$$x_{hk}(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}H_{kn}e^{i\omega_n t_r}$$
$$\dot{x}_{hk}(t_r)=\frac{1}{N}\sum_{n=0}^{N-1}\overline{H}_{kn}e^{i\omega_n t_r} \qquad (r = 0, 1, 2, \ldots, N-1) \tag{22}$$

By using Eq. (3), the Fourier components H_{kn} and $\overline{H}_{kn}$ can be derived as

$$H_{kn} = \sum_{r=0}^{N-1} x_{hk}(t_r) e^{-i\omega_n t_r} = a_k X_{kn} + a_k^* Y_{kn}$$
$$\overline{H}_{kn} = \sum_{r=0}^{N-1} \dot{x}_{hk}(t_r) e^{-i\omega_n t_r} = a_k \lambda_{k1} X_{kn} + a_k^* \lambda_{k1}^* Y_{kn} \tag{23}$$

where

$$X_{kn} = \frac{1-e^{\alpha_{kn} N}}{1-e^{\alpha_{kn}}}, \qquad \alpha_{kn} = (\lambda_{k1} - i\omega_n)\Delta$$
$$Y_{kn} = \frac{1-e^{\beta_{kn} N}}{1-e^{\beta_{kn}}}, \qquad \beta_{kn} = (\lambda_{k1}^* - i\omega_n)\Delta \tag{24}$$

The Fourier components satisfy $H_{k(N-n)} = H_{kn}^*$ and $\overline{H}_{k(N-n)} = \overline{H}_{kn}^*$, where $n = 0, 1, 2, \ldots, N/2$. By collecting $x_{hk}(t)$ and $\dot{x}_{hk}(t)$ given by Eq. (22), one can form the vectors as

$$\boldsymbol{x}_h(t_r) = \frac{1}{N} \sum_{n=0}^{N-1} \boldsymbol{H}_n e^{i\omega_n t_r}$$
$$\dot{\boldsymbol{x}}_h(t_r) = \frac{1}{N} \sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n e^{i\omega_n t_r} \qquad (r = 0, 1, 2, \ldots, N\text{-}1) \tag{25}$$

where

$$\boldsymbol{x}_h = \{x_{h1} \quad x_{h2} \quad \cdots \quad x_{hm}\}^T$$
$$\dot{\boldsymbol{x}}_h = \{\dot{x}_{h1} \quad \dot{x}_{h2} \quad \cdots \quad \dot{x}_{hm}\}^T$$
$$\boldsymbol{H}_n = \{H_{1n} \quad H_{2n} \quad \cdots \quad H_{mn}\}^T$$
$$\overline{\boldsymbol{H}}_n = \{\overline{H}_{1n} \quad \overline{H}_{2n} \quad \cdots \quad \overline{H}_{mn}\}^T \tag{26}$$

By using Eq. (25), the solution of Eq. (14) and its time derivative can be expressed as

$$\boldsymbol{u}_h(t_r) = \frac{1}{N} \boldsymbol{\Phi} \sum_{n=0}^{N-1} \boldsymbol{H}_n e^{i\omega_n t_r}$$
$$\dot{\boldsymbol{u}}_h(t_r) = \frac{1}{N} \boldsymbol{\Phi} \sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n e^{i\omega_n t_r} \qquad (r = 0, 1, 2, \ldots, N\text{-}1) \tag{27}$$

Substituting the total dynamic response Eq. (7), determined from Eqs. (11) and (27) into the initial conditions Eq. (6) yields

$$\boldsymbol{u}_0 = \boldsymbol{u}_f(0) + \boldsymbol{u}_h(0) = \boldsymbol{u}_h(0) = \frac{1}{N} \boldsymbol{\Phi} \sum_{n=0}^{N-1} \boldsymbol{H}_n$$
$$\dot{\boldsymbol{u}}_0 = \dot{\boldsymbol{u}}_f(0) + \dot{\boldsymbol{u}}_h(0) = \dot{\boldsymbol{u}}_h(0) = \frac{1}{N} \boldsymbol{\Phi} \sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n \tag{28}$$

where

$$\boldsymbol{H}_n = \boldsymbol{X}_n \boldsymbol{a} + \boldsymbol{Y}_n \boldsymbol{a}^*, \quad \overline{\boldsymbol{H}}_n = \boldsymbol{\Lambda}_1 \boldsymbol{X}_n \boldsymbol{a} + \boldsymbol{\Lambda}_I^* \boldsymbol{Y}_n \boldsymbol{a}^* \tag{29}$$

with

$$\boldsymbol{a} = \{a_1 \quad a_2 \quad \cdots \quad a_m\}^T$$
$$\boldsymbol{X}_n = diag[X_{kn}], \quad \boldsymbol{Y}_{kn} = diag[Y_{kn}]$$
$$\boldsymbol{\Lambda}_{kn} = diag[\Lambda_{kn}] \tag{30}$$

In Eq. (28), $\boldsymbol{u}_f(0) = 0$ and $\dot{\boldsymbol{u}}_f(0) = 0$ are enforced, because $\boldsymbol{u}_f(t)$ is the forced vibration response part determined to satisfy null initial conditions, which is different from the previous works [14].

Substituting Eq. (29) into Eq. (28) gives

$$\overline{\boldsymbol{X}} \boldsymbol{a} + \overline{\boldsymbol{Y}} \boldsymbol{a}^* = \boldsymbol{d}$$
$$\boldsymbol{\Lambda}_1 \overline{\boldsymbol{X}} \boldsymbol{a} + \boldsymbol{\Lambda}_1^* \overline{\boldsymbol{Y}} \boldsymbol{a}^* = \boldsymbol{v} \tag{31}$$

where

$$\overline{\boldsymbol{X}} = \sum_{n=0}^{N-1} \boldsymbol{X}_n, \qquad \overline{\boldsymbol{Y}} = \sum_{n=0}^{N-1} \boldsymbol{Y}_n$$
$$\boldsymbol{d} = N \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{u}_0, \quad \boldsymbol{v} = N \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \dot{\boldsymbol{u}} \tag{32}$$

Solving Eq. (31) for constants vector $\boldsymbol{a}$ or $\boldsymbol{a}^*$ gives

$$\boldsymbol{a} = \frac{i}{2} \boldsymbol{R} \left(\boldsymbol{\Lambda}_1^* \boldsymbol{d} - \boldsymbol{v} \right) \tag{33}$$

where

$$\boldsymbol{R} = diag[R_k] \tag{34}$$

with

$$R_k = \frac{1}{Imaginary(\lambda_{k1}) \overline{X}_k}, \quad \overline{X}_k = \sum_{n=0}^{N-1} X_{kn} \quad (k = 1, 2, \ldots, m) \tag{35}$$

Once $\boldsymbol{a}$ is computed from Eq. (33) by using given initial conditions, the Fourier components $\boldsymbol{H}_n$ are computed first from Eq. (29) and then use the IFFT to compute

$$\boldsymbol{u}_h(t) \Leftarrow \boldsymbol{\Phi}\, IFFT\{\boldsymbol{H}_n\} \tag{36}$$

As the final step, the total dynamic response can be obtained by summing the forced vibration response part $\boldsymbol{u}_f(t)$ from Eq. (11) and the free vibration response part $\boldsymbol{u}_h(t)$ from Eq. (36).

5.2. Non-proportional damping

When $\boldsymbol{C}$ is the non-proportional damping matrix, Eq. (14) cannot be decoupled anymore by using the modal decomposition method. Thus assume the solution of Eq. (14) in the form

$$\boldsymbol{u}_h(t) = \boldsymbol{A} e^{\lambda t} \quad \text{or} \quad u_{hk}(t) = a_k e^{\lambda t} \quad (k = 1, 2, \ldots, m) \tag{37}$$

Substituting Eq. (37) into Eq. (14) gives

$$\left(\lambda^2 \boldsymbol{M} + \lambda \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{A} = \boldsymbol{0} \tag{38}$$

or

$$\begin{bmatrix} m_{11}\lambda^2+c_{11}\lambda+k_{11} & \cdots & m_{1m}\lambda^2+c_{1m}\lambda+k_{1m} \\ \vdots & \ddots & \vdots \\ m_{m1}\lambda^2+c_{m1}\lambda+k_{m1} & \cdots & m_{mm}\lambda^2+c_{mm}\lambda+k_{mm} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_m \end{Bmatrix} = \{\mathbf{0}\} \tag{39}$$

For the existence of non-trivial solution $\boldsymbol{A}$, one can derive a $2m$-degree algebraic equation with λ as the eigenvalue. In general the eigenvalues solved from the algebraic equation are of complex form. As discussed in reference [12], the eigenvlues will appear in the complex conjugate pairs for the underdamped system, because all the coefficients of the algebraic equation are real. Thus, without loss of generality, the eigenvalues of Eq. (38) can be written in a complex conjugate form as

$$\begin{aligned} \lambda_j &= \xi_j + i\Omega_j \\ \lambda_j^* &= \xi_j - i\Omega_j \end{aligned} \qquad (j = 1, 2, \ldots, m) \tag{40}$$

where Ω_j represents the natural frequency and ξ_j the rate of exponential decay of the jth damped vibration mode.

By substituting the jth eigenvalue $\lambda = \lambda_j$ into Eq. (38), the corresponding eigenvector $\boldsymbol{A}$ can be computed. The ratio between the components a_k of the jth eigenvector are given by [12, 13]

$$\frac{a_1}{C_{j1}} = \frac{a_2}{C_{j2}} = \cdots = \frac{a_m}{C_{jm}} = z_j \tag{41}$$

where C_{jk} is the co-factor of the jth row of the determinant in Eq. (38) for a particular λ_j, and z_j is an arbitrary complex number. From Eq. (37), the kth component of the jth vibration mode corresponding to the jth eigenvalue λ_j is given by

$$u_{hk}^j = a_k e^{\lambda_j t} = z_j C_{jk} e^{\lambda_j t} \quad (k = 1, 2, \ldots, m) \tag{42}$$

Since there are $2m$ eigenvalues λ_j (j = 1, 2, …, $2m$), the kth component of the free vibration response part $\boldsymbol{u}_h(t)$ can be compounded from Eq. (37) as

$$u_{hk}(t) = \sum_{j=1}^{2m} b_j u_{hk}^j = \sum_{j=1}^{m} B_j C_{jk} e^{\lambda_j t} + \sum_{j=1}^{m} B_j^* C_{jk}^* e^{\lambda_j^* t} \tag{43}$$

where the fact that the $2m$ eigenvalues are in the m-pairs of complex conjugate is used. The constants B_j are determined by initial conditions. The time derivative of Eq. (43) is given by

$$\dot{u}_{hk}(t) = \sum_{j=1}^{m} B_j C_{jk} \lambda_j e^{\lambda_j t} + \sum_{j=1}^{m} B_j^* C_{jk}^* \lambda_j^* e^{\lambda_j^* t} \quad (k = 1, 2, \ldots, m) \tag{44}$$

The functions $u_{hk}(t)$ and $\dot{u}_{hk}(t)$ can be represented into the spectral forms as

$$\begin{aligned} u_{hk}(t_r) &= \frac{1}{N}\sum_{n=0}^{N-1} H_{kn} e^{i\omega_n t_r} \\ \dot{u}_{hk}(t_r) &= \frac{1}{N}\sum_{n=0}^{N-1} \overline{H}_{kn} e^{i\omega_n t_r} \end{aligned} \qquad (r = 0, 1, \cdots, N-1) \tag{45}$$

By using the DFT theory, H_{kn} and $\overline{H}_{kn}$ can be expressed as

$$\begin{aligned} H_{kn} &= \sum_{r=0}^{N-1} u_{hk}(t_r) e^{-i\omega_n t_r} \\ \overline{H}_{kn} &= \sum_{r=0}^{N-1} \dot{u}_{hk}(t_r) e^{-i\omega_n t_r} \end{aligned} \tag{46}$$

Substituting Eqs. (43) and (44) into Eq. (46) gives

$$\begin{aligned} H_{kn} &= \sum_{j=1}^{m} \left(B_j X_{jkn} + B_j^* Y_{jkn}\right) \\ \overline{H}_{kn} &= \sum_{j=1}^{m} \left(B_j \lambda_j X_{jkn} + B_j^* \lambda_j^* Y_{jkn}\right) \end{aligned} \tag{47}$$

where

$$\begin{aligned} X_{jkn} &= C_{jk} \frac{1 - e^{(\lambda_j - i\omega_n)\Delta N}}{1 - e^{(\lambda_j - i\omega_n)\Delta}} \\ Y_{jkn} &= C_{jk}^* \frac{1 - e^{(\lambda_j^* - i\omega_n)\Delta N}}{1 - e^{(\lambda_j^* - i\omega_n)\Delta}} \end{aligned} \tag{48}$$

Substitute Eq. (47) into Eq. (45) and collect $u_{hk}(t)$ and $\dot{u}_{hk}(t)$ (k = 1, 2, …, m) to form vectors as

$$\begin{aligned} \boldsymbol{u}_h(t_r) &= \frac{1}{N}\sum_{r=0}^{N-1} \boldsymbol{H}_n e^{i\omega_n t_r} \\ \dot{\boldsymbol{u}}_h(t_r) &= \frac{1}{N}\sum_{r=0}^{N-1} \overline{\boldsymbol{H}}_n e^{i\omega_n t_r} \end{aligned} \tag{49}$$

where

$$\begin{aligned} \boldsymbol{H}_n &= \{H_{1n} \quad H_{2n} \quad \cdots \quad H_{mn}\}^{\mathrm{T}} \\ \overline{\boldsymbol{H}}_n &= \{\overline{H}_{1n} \quad \overline{H}_{2n} \quad \cdots \quad \overline{H}_{mn}\}^{\mathrm{T}} \end{aligned} \tag{50}$$

Now consider the initial conditions. Substituting the total dynamic response Eq. (3) determined by Eqs. (11) and (49) into the initial conditions Eq. (6) gives

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{u}_f(0) + \boldsymbol{u}_h(0) = \boldsymbol{u}_h(0) = \frac{1}{N}\sum_{n=0}^{N-1} \boldsymbol{H}_n \\ \dot{\boldsymbol{u}}_0 &= \dot{\boldsymbol{u}}_f(0) + \dot{\boldsymbol{u}}_h(0) = \dot{\boldsymbol{u}}_h(0) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n \end{aligned} \tag{51}$$

In above, $\boldsymbol{u}_f(0) = 0$ and $\dot{\boldsymbol{u}}_f(0) = 0$ are enforced, because $\boldsymbol{u}_f(t)$ is the forced vibration response part determined to satisfy null initial conditions, which is different from the previous work [15].

By using Eq. (47), Eq. (50) can be rewritten in the form as

$$\begin{aligned} \boldsymbol{H}_n &= \boldsymbol{X}_n \boldsymbol{B} + \boldsymbol{Y}_n \boldsymbol{B}^* \\ \overline{\boldsymbol{H}}_n &= \boldsymbol{X}_n \boldsymbol{\Lambda} \boldsymbol{B} + \boldsymbol{Y}_n \boldsymbol{\Lambda}^* \boldsymbol{B}^* \end{aligned} \tag{52}$$

where

$$\Lambda = diag\left[\lambda_k\right]$$

$$X_n = \begin{bmatrix} X_{11n} & X_{21n} & \cdots & X_{m1n} \\ X_{12n} & X_{22n} & \cdots & X_{m2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1mn} & X_{2mn} & \cdots & X_{mmn} \end{bmatrix} \tag{53}$$

$$Y_n = \begin{bmatrix} Y_{11n} & Y_{21n} & \cdots & Y_{m1n} \\ Y_{12n} & Y_{22n} & \cdots & Y_{m2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1mn} & Y_{2mn} & \cdots & Y_{mmn} \end{bmatrix}$$

Substituting Eq. (52) into Eq. (51) gives

$$\overline{X}B + \overline{Y}B^* = d, \quad \widetilde{X}B + \widetilde{Y}B = v \tag{54}$$

where

$$\overline{X} = \sum_{n=0}^{N-1} X_n, \quad \overline{Y} = \sum_{n=0}^{N-1} Y_n \tag{55}$$

$$\widetilde{X} = \overline{X}\Lambda, \qquad \widetilde{Y} = \overline{Y}\Lambda^*$$

$$d = N\, u_0, \quad v = N\, \dot{u}_0$$

The constants vectors B can be solved from Eq. (35) as

$$B = \left(\widetilde{X} - \widetilde{Y}\,\overline{Y}^{-1}\overline{X}\right)^{-1}\left(v - \widetilde{Y}\,\overline{Y}^{-1} d\right) \tag{56}$$

Once the constants vector B is computed from Eq. (56) by using given initial conditions, the Fourier components H_n are computed first from Eq. (52). Then one can readily compute the free vibration response part $u_h(t)$ by using the IFFT algorithm as follows:

$$u_h(t) \Leftarrow IFFT\{H_n\} \tag{57}$$

6. Numerical examples and discussion

For the evaluation of the present FFT-based SAM, we consider 3-DOFs dynamic systems with two different types of viscous damping. The first type is the proportional damping, for which the modal analysis method can be readily applied to get the analytical solutions, and the second type is the non-proportional damping.

(a) Case 1: proportional damping

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m/2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 3c & -2c & 0 \\ -2c & 3c & -c \\ 0 & -c & 3c \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}$$
$$+ \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix} \tag{58}$$

(b) Case 2: non-proportional damping

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m/2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 3c & -2c & 2c \\ -2c & 3c & -c \\ 2c & -c & 3c \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}$$
$$+ \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 3k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix} \tag{59}$$

where the mass matrix, damping matrix, and stiffness matrix are determined by using the values given by m = 20 kg, c = 40 N·s/m, and k = 2 kN/m. The excitation force is determined by $f(t) = 8\,[1 - s(t-1)]$ N, where $s(t)$ represents the unit step function. The initial conditions are given by

$$\{u_1, u_2, u_3\} = \{1, 0, 0\}\,(\text{mm})$$
$$\{\dot{u}_1, \dot{u}_2, \dot{u}_3\} = \{0, 1, 0\}\,(\text{mm/s}) \tag{60}$$

Figure 1 compares the dynamic responses $u_1(t)$, $u_2(t)$, and $u_3(t)$, obtained by three different solution methods for the dynamic system with proportional damping (i.e., Case 1): the present SAM (circle), the modal analysis method (square) and the 4th order Runge-Kutta method (cross). It is quite straightforward to apply the modal analysis method to obtain exact analytical solutions for the linear dynamic systems with proportional damping. Thus the sufficiently accurate dynamic responses obtained by using the modal analysis method are considered as the reference solutions to evaluate the solutions obtained by the present SAM. The DFT period T = 4 seconds and the number of samples $N = 2^{11}$ are used for the present SAM to obtain the dynamic responses within 0.1% time averaged error with respect to the exact reference solutions, whereas the time increment Δt = 0.00195 seconds is used for Runge-Kutta method. The present SAM is found to provide very accurate solutions which are very close to the exact reference solutions and also to the numerical solutions obtained by Runge-Kutta method.

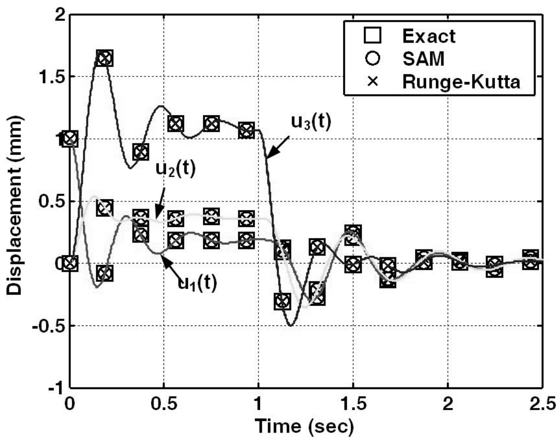

Fig. 1. Proportional viscous damped dynamic responses $u_1(t)$, $u_2(t)$, and $u_3(t)$ obtained by the present SAM (circle), modal analysis method (square) and Runge-Kutta method (cross)

Figure 2 compares the dynamic responses $u_1(t)$, $u_2(t)$, and $u_3(t)$, obtained by the present SAM (circle) and the 4th order Runge-Kutta method (cross) for the dynamic system with non-proportional damping (Case 2). Notice that the dynamic responses obtained by the modal analysis method are not displayed in Fig. 2 because the conventional modal analysis method can not be directly applied to the dynamic systems with non-proportional damping such as for the Case 2. One may find from Fig. 2 that the present SAM certainly provides the dynamic responses which are very close to those obtained by Runge-Kutta method.

Figure 3 shows the convergence of the dynamic responses of the dynamic system with non-proportional damping obtained by the present SAM as the number of samples (N) is increased. As expected, Fig. 3 shows that more accurate solutions can be achieved by increasing N for a fixed time window T = 4 seconds. Very accurate dynamic responses within 0.1% time averaged difference from those obtained by Runge-Kutta method are obtained by using $N = 2^{12}$ for the present example problem. Finally, it is worthwhile to confirm from Fig. 1 through Fig. 3 that the present SAM exactly captures all non-zero initial conditions in the dynamic responses.

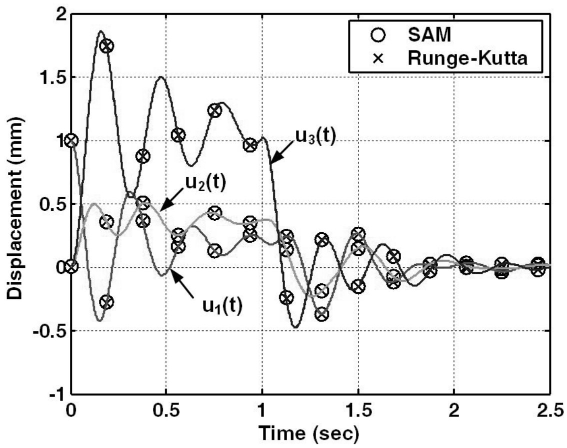

Fig. 2. Non-proportional viscous damped dynamic responses $u_1(t)$, $u_2(t)$, and $u_3(t)$ obtained by the present SAM (circle) and Runge-Kutta method (cross)

7. Conclusions

This paper proposes an FFT-based SAM for the dynamic responses of viscously damped linear discrete dynamic systems which are subjected to arbitrary initial conditions. In this paper, the total dynamic response is assumed as the sum of the forced vibration response part determined purely by the excitation force to satisfy null initial conditions and the free vibration response part determined purely by only non-zero initial conditions, which is basically different from the previous works [14] and [15]. Through some numerical examples, it is shown that the present FFT-based SAM can provide sufficiently accurate solutions by choosing a proper FFT conditions such as the sampling number and time window. The high accuracy of the present FFT-based SAM is proved by comparing its solutions with those obtained by the modal analysis method and Runge-Kutta method.

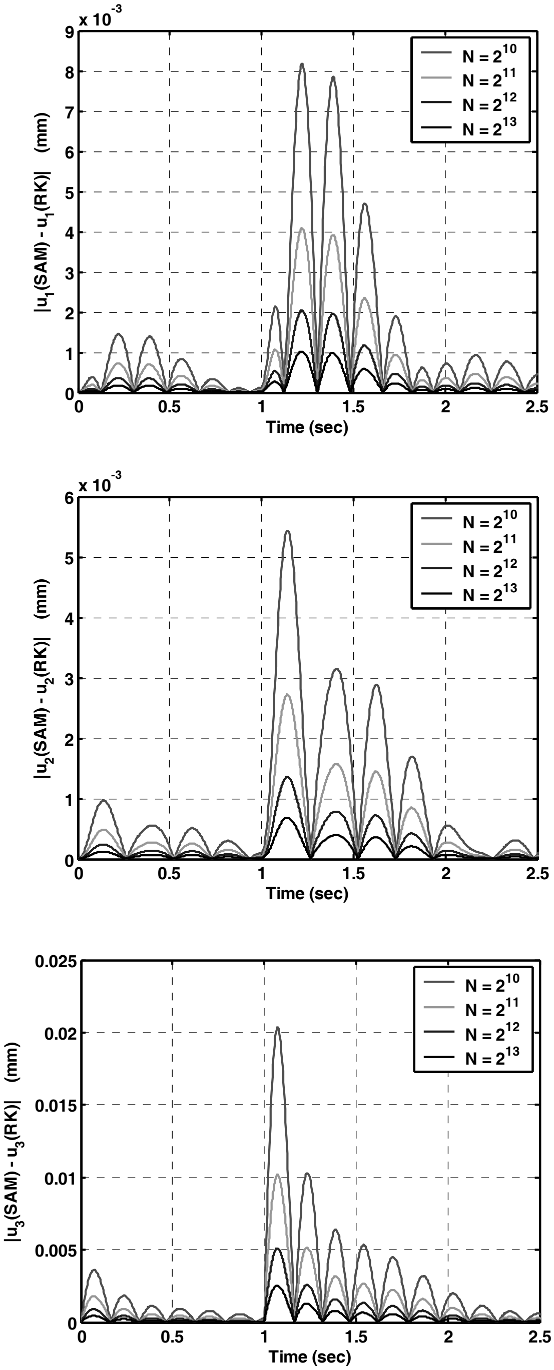

Fig. 3. Non-proportional viscous damped dynamic responses obtained by the present SAM vs. the sampling number N

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