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Application of the computer tomography in examination of the internal structure of materials by considering the specific con-ditions of the problem

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Analysis and modelling

ABSTRACT

Purpose of this paper: In this paper we present a summary of the results reached in the field of computer tomography applied in some special case – for the problem of incomplete projection data. This particular problem arises in the technical issues in which, for some reasons (like for example size of the examined object, its localization or its accessibility), it is impossible to apply the standard algorithms of computer tomography.

Design/methodology/approach: In the paper we discuss the standard algebraic algorithms of computer tomography and, additionally, the new algebraic algorithms (parallel and chaotic), designed by the authors, suitable not only for the case of incomplete projection data but also useful in the standard approach.

Findings: The above mentioned algorithms are tested in solving the problems of reconstruction the discrete objects of high-contrast. Moreover, convergence, stability and utility of the algorithms are proved experimentally.

Research limitations/implications: Algorithms, created by the authors, are designed for the multiprocessor computers which allow to execute the calculations simultaneously. However, the results compiled in the paper were elaborated by using the one-processor computer. Calculations in which the parallel computing structure will be used are planned for the nearest future.

Practical implications: Possibilities of the effective applications of the discussed algorithms in different practical technical problems are showed in the paper. Research, done till now, indicate the chances of applying the proposed algorithms in certain technical problem in which the incomplete projection data appear (like, for example, in searching for the elements in material which cause decreasing of its strength or in looking for the compressed gas reservoirs in the coal bed, which can be dangerous for the people's life and health).

Originality/value: The paper presents the reconstruction algorithms (block and chaotic-bloc), designed by the authors, which appear to be more effective than the standard algebraic algorithms adapted for solving problems with the incomplete projection data.

Keywords: Numerical techniques; Computer tomography; Parallel algorithms; Chaotic algorithms

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1. Introduction

Computer tomography can find an application not only in medicine but also in solving a wide class of technical problems. Methods of computer tomography can be used in every case in which the examination of internal structure of an object, without its destruction, is needed. Algorithms of computer tomography can be divided into two groups: analytic and algebraic algorithms. For some reasons we will apply only the algebraic ones.

Let $f(x,y)$ be a function which represents the spatial distribution of some physical parameter. If L is a line (ray) in the plane, then the line integral:

$$p_L = \int_L f(x,y)dL, \qquad (1)$$

which is called a projection, is usually obtained from the physical measurements.

From the mathematical point of view, the problem of reconstruction from projections consists in finding an unknown function $f(x,y)$ by means of the given set of projections p_L, for all rays L. Theoretically, it is possible to reconstruct the function $f(x,y)$ from the set p_L by means of the Radon inversion formula. However, in practice we have given only the discrete set of projection data that estimate p for a finite number of rays. Moreover, since the projection data are obtained by physical measurements, they contain some errors.

In many practical applications, the projection data are usually not available at each direction and, additionally, may be very limited in number. In this case we say that we have a problem of image reconstruction with incomplete projection data. In particular, such kind of problem arises in mineral industries and engineering geophysics connected with acid drainage, the stability of mine workers, mineral exploration and others [1,2].

In the algebraic algorithms, after the discretization of the considered region, it is assumed that the energy lost by the given ray is equal to the sum of energies lost in the particular pixels occurring in the trajectory of this ray, and that every pixel absorbs the portion of energy which is proportional to the value of function f in this pixel and to the length of path passed by the ray to this pixel. Values of the absorption coefficients are unknown, whereas regions of the intersections of rays and pixels can be determined by knowing the discretization density and the equations of lines containing those rays. There are also known the initial and terminal values of the rays energies, so in consequence difference between them, which means that all of the projection values are known. Those pieces of information give the basis for formulating the system of linear equations. Problem of computer tomography, determined in this way, consists in solving the following system of linear equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{p}, \qquad (2)$$

where we use the following notation:

$\mathbf{A} = (a_{ij}) \in \mathbf{R}^{m,n}$ – matrix of coefficients;

$\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbf{R}^n$ – vector of unknown elements;

$\mathbf{p} = (p_1, p_2, ..., p_m)^T \in \mathbf{R}^m$ – vector of projection.

Method of solution of the above system of equations is equivalent to applying the considered algebraic algorithms.

2. Algorithms ART and ART-3

We consider the system (2) of algebraic linear equations, constructed in the way described in the previous section. For solving this problem we describe two algorithms.

2.1. Algorithm ART [3-5]

Let us introduce the following notations:

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^i, \mathbf{x}) - p_i}{\|\mathbf{a}^i\|^2}\mathbf{a}^i, \qquad (3)$$

$$\mathbf{P}_i^{\omega} = (1-\omega)\mathbf{I} + \omega\mathbf{P}_i, \qquad (4)$$

where $\mathbf{a}^i$ is the *i-th* row of the matrix $\mathbf{A}$, $0 < \omega < 2$ denotes the relaxation parameter and $\mathbf{I}$ refers to the identity matrix. Then the algorithm runs as follows:

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. *(k+1)-th* vector is calculated in accordance of the formula:

$$\mathbf{x}^{(k+1)} = \mathbf{P}_i^{\omega_k}\mathbf{x}^{(k)}, \quad for \quad i = 1,...m, \qquad (5)$$

where: $\mathbf{P}_i^{\omega_k}$ is an operator defined by means of (4), ω_k denotes the relaxation parameter and $i(k)=k(\text{mod } m)+1$. The convergence conditions of the ART algorithm are proved in [5].

2.2. Algorithm ART [3-5]

Let us denote:

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{((\mathbf{a}^i, \mathbf{x}) - p_i - \varepsilon_i)^+ - (p_i - \varepsilon_i - (\mathbf{a}^i, \mathbf{x}))^+}{\|\mathbf{a}^i\|^2}\mathbf{a}^i, \qquad (6)$$

where the following symbol appears:

$$s^+ = \begin{cases} s, & \text{if } s \geq 0; \\ 0, & \text{otherwise} \end{cases} \qquad (7)$$

and:

$$\mathbf{P}_i^{\omega} = (1-\omega)\mathbf{I} + \omega\mathbf{P}_i, \qquad (8)$$

where: $\mathbf{a}^i$ is the *i-th* row of the matrix **A**, $0 < \omega < 2$ denotes the relaxation parameter and **I** refers to the identity matrix. Then the algorithm runs as follows:
1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. *(k+1)-th* vector is obtained by the formula:

$$\mathbf{x}^{(k+1)} = \mathbf{P}_i^{\omega_k}\mathbf{x}^{(k)}, \quad for \quad i = 1,...m, \tag{9}$$

where: $\mathbf{P}_i^{\omega_k}$ is an operator defined by means of relations (6) and (8), ω_k denotes the relaxation parameter and $i(k)=k(\text{mod } m)+1$. In this case, vector $\mathbf{e} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ refers to the vector of errors which noise the projections **p**. Then, instead of solving the system of equations (2) we solve the system of inequalities of the form:

$$\mathbf{p} - \mathbf{e} \le \mathbf{A} \cdot \mathbf{x} \le \mathbf{p} + \mathbf{e}.$$

3. Parallel and chaotic algorithms [6-17]

Algorithms ART and ART-3 are useful in solving the standard problems as well as in considering the problems of incomplete projection data. However, in cases of the significant data shortage the convergence of described algorithms becomes slow. In this section we present some already well-known algorithms and we introduce algorithms designed by the authors for the purpose of speeding up the reconstruction of the examined objects. In the previous considerations, the selection order of the equations in the successive iterations of the algorithms was always the same. It turns out that the order of selection can have a big influence for the speed of algorithm convergence. Algorithms of that kind are the asynchronous algorithms, which include the chaotic algorithms. Another approach for increasing the speed of running of the algorithm consists in introducing the special type of parallel algorithms and implementing them in the parallel computing systems. Such kind of process happens by including, to the operation of solving the system of equations, some group of processors working independently and simultaneously, which can significantly reduce the time of determining single iteration. Group of algorithms realising this idea are the block-parallel algorithms. Parallelism of the algorithm can be obtained in two ways. After dividing the matrix of coefficients into blocks, every block corresponds with one processor, which uses only the rows of matrix contained in this particular block and generates the partial solution. In the next step, the central processor averages the solution, which ends the iteration. Another way for receiving parallelism of the algorithm consists in, similarly, dividing the coefficient matrix into blocks corresponding with the processors, working independently and simultaneously, but the operations into every single block are executed sequentially. Every successive solution is received as the averaged value of solutions of all blocks.

3.1. Iterative-block algorithms

In practical realization of the parallel algorithms a big number of local processors in the parallel computing structures is required. For the purpose of reducing the number of required local processors we will consider the block-iterative algorithms.

Let us decompose the matrix **A** and the projection vector **p** into *M* subsets according to the condition:

$$\{1,2,...,m\} = H_1 \cup H_2 \cup ... \cup H_M, \tag{10}$$

where $\{1,2,...,m\}$ is the set of indices of the matrix rows and:

$$H_t = \{m_{t-1}+1, m_{t-1}+2, ..., m_t\}, \tag{11}$$

for $0 = m_0 < m_1 < ... < m_M = m$.

In the, new introduced, iterative-block algorithm SZB-3 we have the following steps:
1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. *(k+1)-th* vector is received in accordance of the formula:

$$\mathbf{x}^{(k+1)} = \mathbf{C}\sum_{i\in H_{t(k)}} \mathbf{B}_i^k \mathbf{P}_i^{\varpi_k}\mathbf{x}^{(k)}, \tag{12}$$

where $t(k) = k(\text{mod } M) + 1$, $\mathbf{P}_i^{\omega_k}$ is an operator defined with the aid of formulas (6) and (8), $0 < \omega_k < 2$ denotes the relaxation coefficient, **C** is the constraining operator (defined in section 3.3) and $\mathbf{B}_i^k$ describes the matrix of dimension $n \times n$, with the nonnegative elements of the form:

$$\mathbf{B}_i^k = diag\{b_1^{k,i}, b_2^{k,i}, ..., b_n^{k,i}\}, \tag{13}$$

where:

$$b_p^{k,i} = \frac{y_p^{k,i}}{\sum_{i\in H_{t(k)}} y_p^{k,i}}, \tag{14}$$

for $p = 1,2,...,n.$

3.2. Parallel-block algorithms

Algorithms considered in the previous sections represent the algorithms in which the parallel work runs in every block, whereas the blocks are connected sequentially. In the algorithms presented in this section, operations are executed sequentially in blocks, while the blocks work simultaneously.

Let the matrix **A** and the projection vector **p** be decomposed into blocks, according to the formulas (10) and (11). For every block H_i we introduce an operator, denoted by $\mathbf{Q}_i$, defined by means of composing the operators $\mathbf{P}^{\varpi}_{m_i}, \mathbf{P}^{\varpi}_{m_i-1}, ..., \mathbf{P}^{\varpi}_{m_{i-1}+1}$, determined by the conditions (6) and (8), indices of which belong to the block H_i:

$$\mathbf{Q}_i = \mathbf{P}^{\varpi}_{m_i}\mathbf{P}^{\varpi}_{m_i-1}...\mathbf{P}^{\varpi}_{m_{i-1}+1}, \quad (15)$$

The new designed parallel-block algorithm RB-3 runs as follows:

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. *(k+1)-th* vector is received by the formula:

$$\mathbf{x}^{(k+1)} = \sum_{i=1}^{M} \mathbf{B}_i \mathbf{y}^{k+1,i}, \quad (16)$$

where:

$$\mathbf{y}^{k+1,i} = \mathbf{Q}_i \mathbf{x}^{(k)}, \quad (17)$$

$\mathbf{Q}_i$ is an operator described by the formula (15) and $\mathbf{B}_i$ refer to the matrix of dimension $n \times n$, with the nonnegative elements of the form:

$$\mathbf{B}_i = diag\{b_1^i, b_2^i, ..., b_n^i\}, \quad (18)$$

where:

$$b_p^i = \frac{\sum_{s \in H_i} a_{s,p}}{\sum_{i=1}^{M} a_{s,p}} \quad (19)$$

for $i = 1,2,...,M$ and $p = 1,2,...,n.$

3.3. Chaotic algorithms

From the mathematical point of view, the asynchronous algorithms are based on the methods of asynchronous iterations, which were first introduced by D.Chazan and W.Miranker (Chazan and Miranker, 1969) under the name "random relaxations". The presented methods have found the first application for solving the systems of linear algebraic equations. Further development of these methods and their generalization for the case of nonlinear operators were given by G.M.Baudet (Baudet, 1978). The important results were also obtained by El Tarazi (El Tarazi, 1984), who introduced a visual model for the class of asynchronous algorithms and obtained the first correct conditions of convergence in the nonlinear case for contracting operators.

Let us present few definitions.

Definition A sequence of nonempty subsets $I = \{I_k\}_{k=0}^{\infty}$ of the set {1,2…, m} is a sequence of chaotic sets if $\limsup_{j\to\infty} I_j = \{1,2,...,m\}$. (In other words, if each integer $j \in \{1,2,...,m\}$ appears in this sequence infinite number of times).

Definition If each subset I_k of the sequence of chaotic sets $I = \{I_k\}_{k=0}^{\infty}$ consists of only one element, then such sequence is called acceptable.

Definition A sequence $J = \{\sigma(k)\}_{k=1}^{\infty}$ of *m*-dimensional vectors $\sigma(k) = (\sigma_1(k), \sigma_2(k), ..., \sigma_m(k))$ with the integer coordinates, satisfying the following conditions:

$$0 \le \sigma_i(k) \le k-1 \quad (20)$$

$$\lim_{k\to\infty} \sigma_i(k) = \infty, \quad (21)$$

for each $i = 1,2,..., m$ and $k \in \mathbf{N}$, is called a sequence of delays.

Let $T = \{T_i\}_{i=1}^{m}$ be a set of nonlinear operators, acting in the Euclidean space $\mathbf{R}^n$ and let **S** be an algorithmic operator. We will consider the following iterative process:

$$y^{k,i} = T_i x^{(k-1)}, \quad (22)$$

$$x^k = S(x^{(k-1)}, \{y^{k,i}\}_{i=1}^{m}), \quad (23)$$

where:

x denotes an *n*-dimensional vector of the space $\mathbf{R}^n$ and $i \in \{1,2,...,m\}$, for every k = 0,1,2,.... Then we can formulate one more definition.

Definition Let $\mathbf{T}_i : \mathrm{R}^n \to \mathrm{R}^n$, $i \in \{1,2,...,m\}$ be a set of nonlinear operators and let $\mathbf{x}^{(0)} \in \mathbf{R}^n$ be an initial value of the vector x. A generalized model of the asynchronous iterations with limited delays for the set of operators $\mathbf{T}_i$, *i*=1,2,...,*m,* is a method of building the sequence of vectors $\{\mathbf{x}^k\}_{k=0}^{\infty}$, which is given recursively by the following scheme:

$$\mathbf{y}^{k,i} = \begin{cases} \mathbf{T}_i \mathbf{x}^{(\sigma^i(k))}, & \text{if } i \in I_k \\ \mathbf{y}^{k-1,i}, & \text{otherwise} \end{cases} \quad (24)$$

$$\mathbf{x}^{(k)} = \mathbf{S}\left(\mathbf{x}^{(k-1)}, \left\{\mathbf{y}^{k,i}\right\}_{i \in I_k}\right), \quad (25)$$

where $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$ and $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ refer to the sequences of limited delays, for $i=1,2,...,m$.

Algorithm CHART-3 runs in the following way:
1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. $k+1$-th iteration is calculated in accordance with following scheme:

$$\mathbf{y}^{k,i} = \begin{cases} \mathbf{P}_i^{\omega_k}\mathbf{x}^{(k-1)}, & \text{if } i \in I_k, \\ \mathbf{y}^{k-1,i}, & \text{otherwise,} \end{cases} \tag{26}$$

$$\mathbf{x}^{(k)} = \mathbf{C}\sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k,i}, \quad (i = 1,2,...,m), \tag{27}$$

where $\mathbf{P}_i^{\omega_k}$ are the operators defined by means of (6) and (8), ω_k denote the relaxation parameters with property $0<\omega_k<2$, γ_i^k are the positive real numbers with property:

$$\sum_{i \in I_k} \gamma_k^i = 1, \tag{28}$$

for each $k \in \mathbf{N}$, $I = \{I_k\}_{k=1}^{\infty}$ is the acceptable sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$ and, finally, $\mathbf{C}$ is the constraining operator. In this paper we consider such $\mathbf{C}=C_1\, C_2\, C_3$, where:

$$C_1[\mathbf{x}] = \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \in D, \\ 0, & \text{otherwise,} \end{cases} \tag{29}$$

$$(C_2[\mathbf{x}])_i = \begin{cases} a, & \text{if } x_i < a, \\ x_i, & \text{if } a \le x_i \le b, \\ b, & \text{if } x_i > b, \end{cases} \tag{30}$$

and

$$(C_3[\mathbf{x}])_j = \begin{cases} 0, & \text{if } p_i = 0 \wedge a_{ij} \ne 0, \\ x_j, & \text{otherwise.} \end{cases} \tag{31}$$

3.4. Chaotic-block algorithms

Both of the previous approaches for the problem of reconstruction of the internal structure of the object - chaotic and block algorithms – can be combined together. In this way the chaotic-block algorithms are received.

New introduced algorithm CHRB-3 runs as follows:
1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. $k+1$-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{k+1} = \mathbf{C}\sum_{i=1}^{M} \mathbf{B}_i^k y^{(k+1),i}, \tag{32}$$

in which the following notation is used:

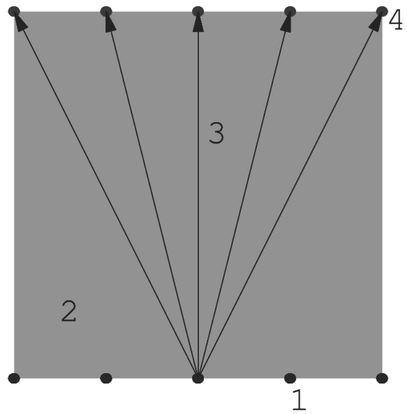
$$\begin{aligned} y^{(k+1),i} &= \mathbf{Q}_i \mathbf{x}^k, \\ \mathbf{Q}_i &= \mathbf{P}_{i,s_i}\mathbf{P}_{i,s_i-1}\ldots\mathbf{P}_{i,1}, \\ \mathbf{P}_{i,j} &= \mathbf{P}_j^{\varpi}, \quad j \in I_{i(j)}, \end{aligned} \tag{33}$$

where $\mathbf{P}_i^{\omega}$ refer to the operators defined by relations (6) and (8), symbol $0 < \omega < 2$ describes the relaxation parameters, $\mathbf{C}$ is the constraining operator, $I = \{I_{i(k)}\}_{k=1}^{\infty}$ denotes the sequence of chaotic sets such that $I_{i(k)} \subset \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\} = H_i$ and $\mathbf{B}_i^k$ are the matrices of dimension $n \times n$, with real nonnegative elements which satisfy conditions (18) and (19), for each $k \in N$.

4. Computer simulation and experimental results

In dependence on the obtaining system of projections there are many image reconstruction schemes. The main of them are parallel and beam schemes in the two-dimensional space. In some practical problems, in engineering for example, it is impossible to get projections from all directions because of some important reasons (such as localization, size or impossibility of an access to the researched object). Such situation arises, for example, in the coal bed working. In the coal bed, during the preparation process for working, the access to longwalls may be very difficult or impossible at all, in dependence on the method of coal mining. Sometimes it is impossible to access to one or two sides of longwalls, and sometimes it is only impossible to access to the basis but all the longwalls are accessible. Each of this situation has its own scheme of obtaining the information. In this paper we present the results of the image reconstructions only for two different, the most natural, schemes of obtaining the projection data, which are described below.

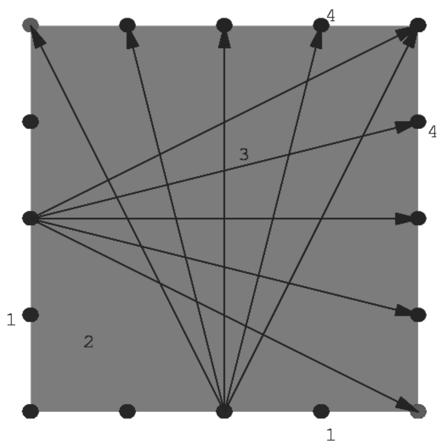
In the first scheme, which we will call as the system (1×1), we have an access to the research object from only two opposite sides. This situation often arises in engineering geophysics. In this case the sources of rays are situated only on one side and the detectors are situated on the opposite side of the researched part of, for example, a coal bed. This scheme of obtaining information is shown in Figure 1.



· Sources of rays
· Research object
· Rays
· Detectors

Fig. 1. Scheme of the system (1×1)

The second scheme of obtaining projection data, called (1×1,1×1), is shown in Figure 2. In this situation we can have an access to all four sides of the examined object. Therefore, the sources can been situated on two neighboring sides, and the detectors can been situated on the opposite sides. In this way, the projections can be obtained from two pairs of the opposite sides.



1 - Sources of rays
2 - Research object
3 - Rays
4 - Detectors

Fig. 2. The system (1×1, 1×1)

In order to evaluate the good quality of the computed reconstruction of a high-contrast image, from the limited number of projections and incomplete data, we have tested different kinds of geometric figures and reconstruction schemes. In the simulation process of image reconstruction an important factor is the choice of the density distribution, discrete or continuous, of the researched object. In a coal bed, where we search the reservoirs of compressed gas or interlayers of a barren rock, the density distribution may be considered as a discrete function and the density difference of these three environments (coal, compressed gas and barren rock) is significant. Therefore, for illustrating the implementation of the algorithms working with incomplete data, we chose the discrete function with high contrast, which is given in the following form:

$$f(x,y)=\begin{cases}1, & (x,y)\in D\subset E\subset \mathbf{R}^2,\\ 0, & \text{otherwise,}\end{cases} \tag{33}$$

where E is a square $E=\{(x,y):-1\le x,y\le 1\}$ and D is a subset of E of the following form:

D = [-0.4,-0.2] x [-0.5,0.5] ∪ [-0.2,0.2] x [0.3,0.5] ∪ ∪ [-0.2,0.2] x [-.1,0.1] ∪ [0,0.2] x [0.1,0.3].

Another form of the discrete function representing the density distribution can be the following:

$$f_1(x,y)=\begin{cases}1, & (x,y)\in D_1\subset E\subset \mathbf{R}^2,\\ 2, & (x,y)\in D_2\subset E\subset \mathbf{R}^2,\\ 3, & (x,y)\in D_3\subset E\subset \mathbf{R}^2,\\ 4, & (x,y)\in D_4\subset E\subset \mathbf{R}^2,\\ 0, & \text{otherwise}\end{cases} \tag{33}$$

where E is a square $E=\{(x,y):-1\le x,y\le 1\}$ and D_i are subsets of E of the form:

D_1 = [-0.7,-0.4] × [-0.5,0.2],
D_2= [-0.2,0.2] × [-0.1,0.1],
D_3 = [-0.2,0.2] × [0.3,0.5],
D_4 = [0.4,0.7] × [0.4,0.7].

In Figures 3-6 plots of the exemplary functions of density distribution (examined in the paper) are presented. Those functions are discrete and of high-contrast. Selection of such kind of functions is not accidental, because in technical problems, in which the proposed algorithms can find an application, the density distribution is also discrete and the differences of density between the particular environments are significant. In Figures 3 and 5 the three-dimensional view of the plot of the density distribution function is displayed, while in Figures 2 and 4 the two-dimensional view is showed

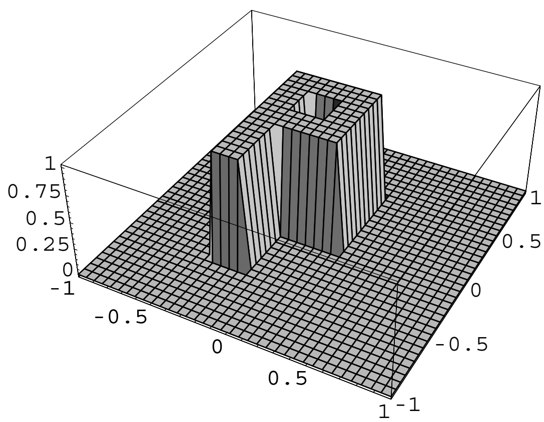


Fig. 3. Three-dimensional view of the plot f

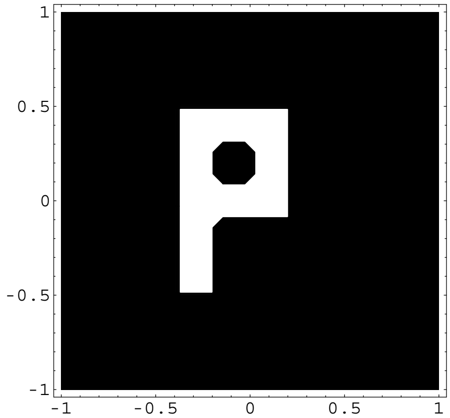


Fig. 4. Two-dimensional view of the plot of f (the black color denotes the value 0 and the white color denotes the value 1)

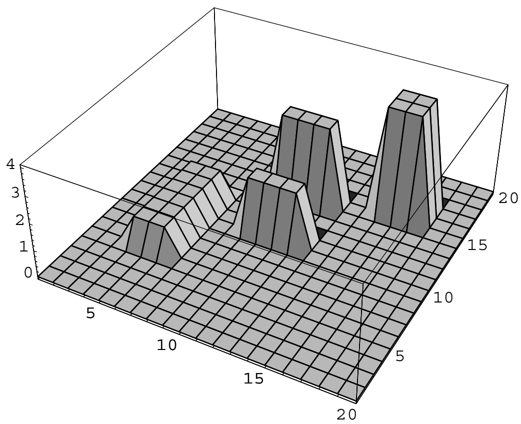


Fig. 5. Three-dimensional view of the plot f_1

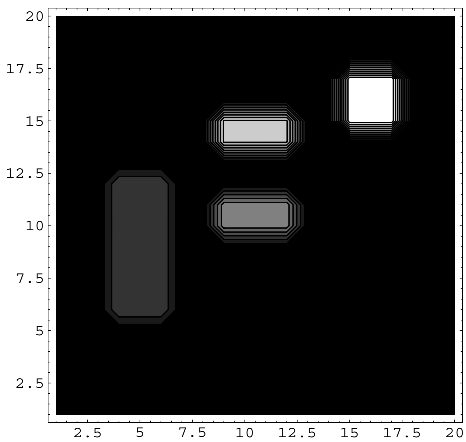


Fig. 6. Two-dimensional view of the plot of f_1

Quality of the received reconstruction will be verified with the aid of the following measures of error - the maximum absolute error:

$$\Delta = \max_i \left| f_i - \tilde{f}_i \right| ,$$

the maximum relative error:

$$\delta\% = \frac{\max_i \left| f_i - \tilde{f}_i \right|}{\max_i (f_i)} \cdot 100\% ,$$

and the mean absolute error:

$$\delta = \frac{1}{n} \sum_i \left| f_i - \tilde{f}_i \right| ,$$

where f_i is the value of the given modeling function in the center of the i-th pixel and $\tilde{f}_i$ is the value of the reconstructed function in the i-th pixel.

The reconstruction result of $f(x,y)$ with the aid of the algorithm ART-3, after 15 iterations in the reconstruction scheme (1x1, 1x1), for n=20 x 20 pixels and for m= 644 projections is presented in Figure 7 (where plot of the reconstruction function is displayed) and in Figure 8 (where plot of the mean absolute error for this image reconstruction is showed). Dependence of the mean absolute error and the maximum relative error on the number of iterations for this case of image reconstruction (received by using algorithms ART-3 and CHART-3) is presented in Figures 9 and 10, respectively.

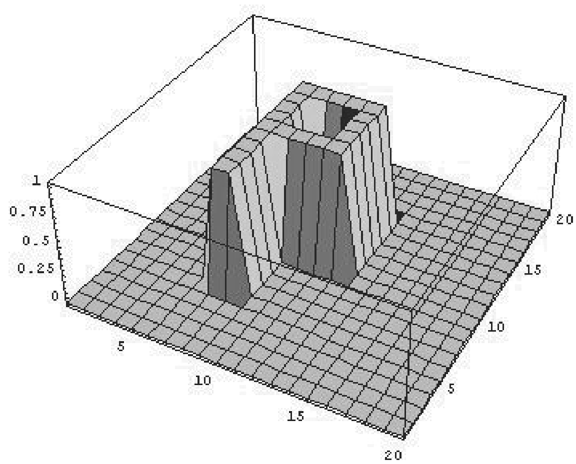


Fig. 7. Plot of the reconstruction function f

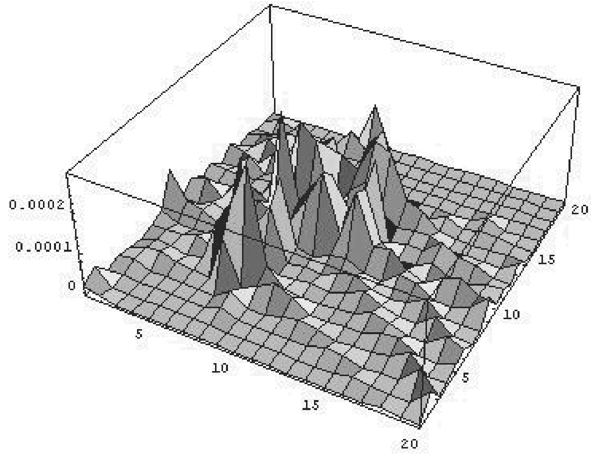


Fig. 8. Plot of the mean absolute errors

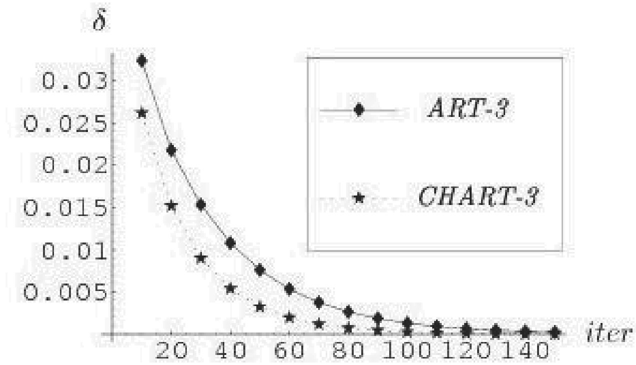


Fig. 9. Dependence of the mean absolute error on the number of iterations for image reconstruction of *f* in the system (1*x*1)

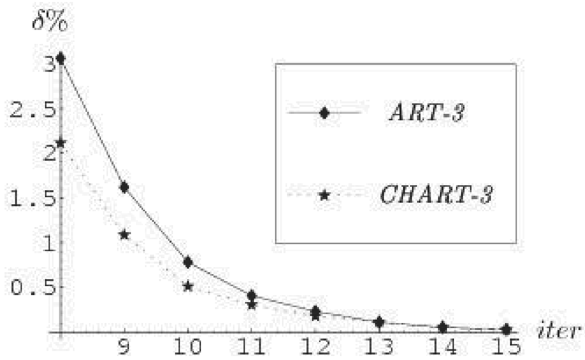


Fig. 10. Dependence of the maximum relative error on the number of iterations for image reconstruction of *f* in the system (1*x*1,1*x*1)

The executed experiments show that the chaotic algorithms are more useful, in case of every reconstructed function and for every set of reconstruction parameters, than their equivalents which does not apply the randomization. The presented research concerned particularly the problem of incomplete projection data oriented towards some technical applications. However, in the standard approach the chaotic algorithms are also more efficient.

Similarly, by comparing the standard algorithms of computer tomography with the block algorithms it turns out that the block algorithms are more useful (considering the time needed for obtaining the required error, not the number of iterations after which the assumed error is received - do not forget that in the block algorithms we use the simultaneous work of many processors).

Considering the comparison between two types of block algorithms we suppose to lean towards the RB-3 algorithm. Advantage of this algorithm results from two facts: firstly, because of the technical reasons - in the SZB-3 algorithm one need to use much bigger number of processors than in RB-3 algorithm, secondly, because speed of convergence of algorithm RB-3 is better than in case of algorithm SZB-3.

In Figure 11 the comparison of the more useful algorithm RB-3 with its equivalent applying the randomization - algorithm CHRB-3 is presented. It can be seen that the RB-3 algorithm is the most useful algorithm among all the considered algebraic algorithms with the incomplete projection data.

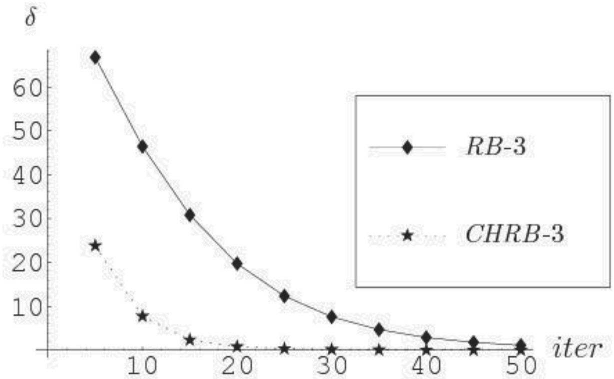


Fig. 11. Dependence of the mean absolute error on the number of iterations for image reconstruction of *f* with algorithm RB-3 and CHRB-3 in the system (1 × 1, 1 × 1)

Table 1 shows the number of iterations needed for obtaining the image reconstruction with the given maximum relative error for each of considered algorithms: ART-3, BP-3, CHRB-3 and for each of considered schemes.

Table 1.
The number of iterations which need for obtaining the required equality - the maximum relative error δ

		< 10%	< 5%	<1%	<0.5%
(1×1, 1×1)	ART-3	8	9	12	14
	RB-3	13	23	47	60
	CHRB-3	24	30	46	53
(1×1)	ART-3	23	37	66	89
	RB-3	74	178	953	1279
	CHRB-3	95	148	271	340

Additional advantage of the examined algorithms is their stability. The stability examining of the considered algorithms consists in including the noises (random vector of the normal or uniform distribution) into the projection vector **p** and testing the influence of this noise on the solution. In Tables 2 and 3 errors of the received reconstructions are compiled.

Table 2.
Table of the reconstruction errors in the simulation of algorithm ART-3, for the resolution 20x20, 18 sources and detectors, with the normal noise of s% , in the system (1x1 1x1)

iter	0.15%	0.75%	1%	2%	5%	
10	0.01268	0.03330	0.04385	0.07668	0.19258	Δ
10	0.00065	0.00221	0.00289	0.00574	0.01449	δ
20	0.00577	0.02887	0.03849	0.07698	0.19245	Δ
20	0.00043	0.00217	0.00289	0.00579	0.01449	δ
75	0.00577	0.02886	0.03849	0.07698	0.19245	Δ
75	0.00043	0.00217	0.00289	0.00579	0.01449	δ

Table 3.
Comparison of errors δ received for 4 simulations *sym* of the algorithms SZB-3, RB-3 and CHRB-3

	sym1	*sym2*	*sym3*	*sym4*	
SZB-3	4.1346	4.5110	3.5822	3.6166	δ%
RB-3	4.3924	3.0255	4.0933	4.4307	δ%
CHRB-3	3.7321	3.6065	3.7341	2.7132	δ%

In Figures 12-18 the image reconstructions and plots of error for the selected density distribution functions and the different reconstruction algorithms are presented.

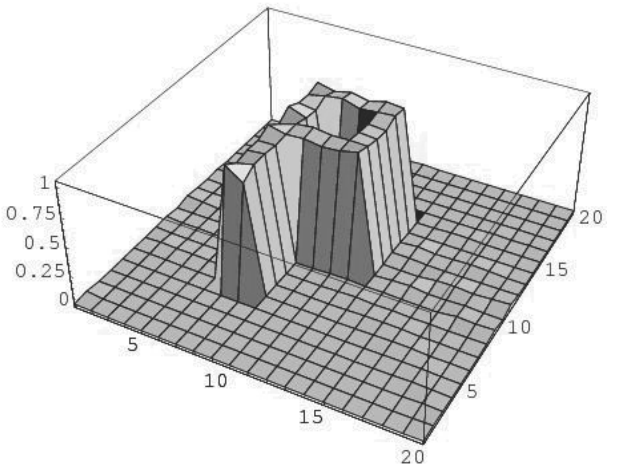


Fig. 12. Three-dimensional plot of the reconstruction in simulation of the algorithm ART-3 for resolution 20x20, 18 sources and detectors, with the normal noise of 2%, in the system (1x1 1x1)

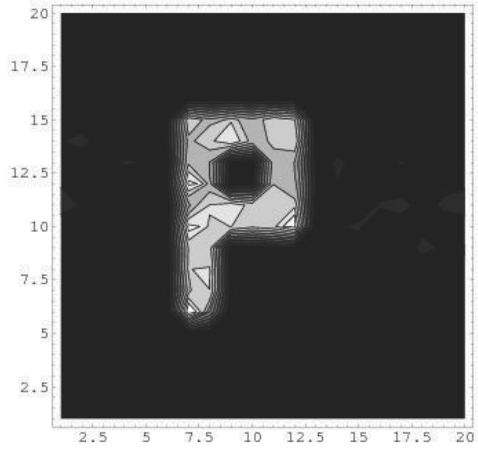


Fig. 13. Contour plot of the reconstruction in the simulation of algorithm ART-3, for resolution 20x20, 18 sources and detectors, with the normal noise of 10%, in the system (1x1 1x1)

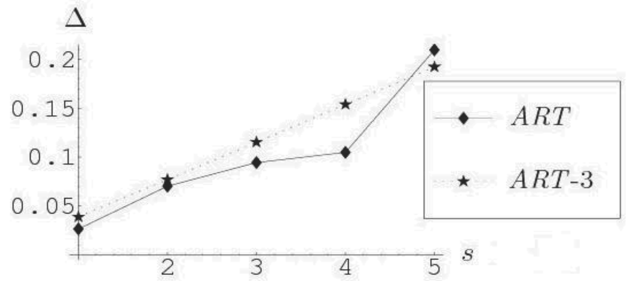


Fig. 14. Comparison of the of s% noise influence on the error Δ in simulation of algorithms ART-3 and ART in the system (1x1)

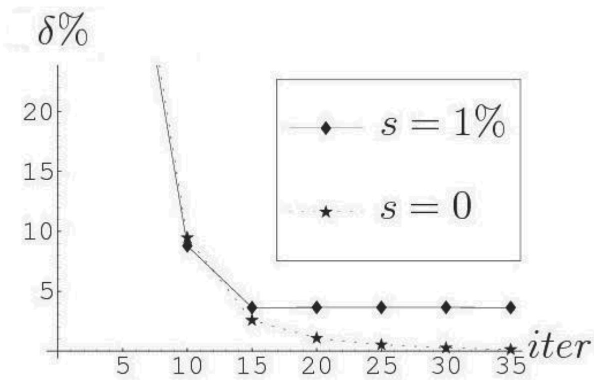


Fig. 15. Dependence of the error δ% on the number of iterations in simulation of the algorithm SZB-3 for resolution 20x20, 18 sources and detectors, with and without the normal noise of s%

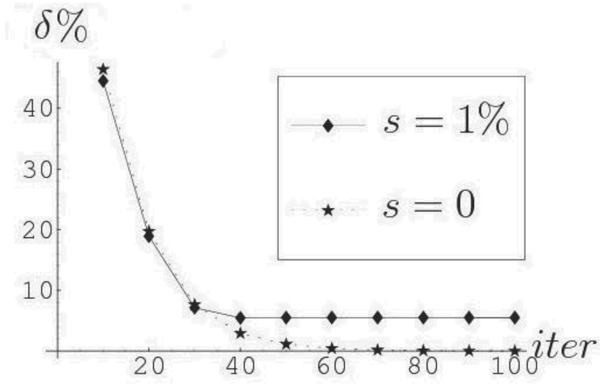


Fig. 16. Dependence of the error $\delta\%$ on the number of iterations in simulation of the algorithm RB-3 for resolution 20x20, 18 sources and detectors, with and without the normal noise of s%

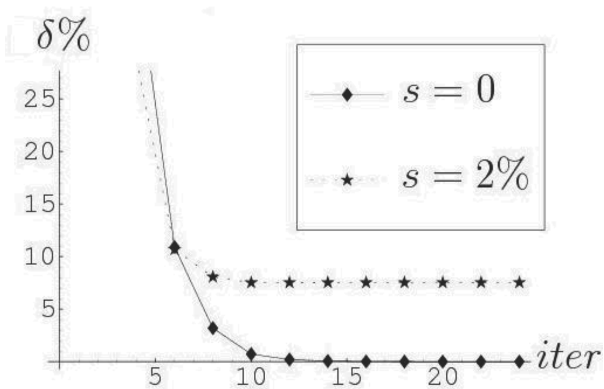


Fig. 17. Dependence of the error $\delta\%$ on the number of iterations in simulation of the algorithm CHART-3 for resolution 20x20, 18 sources and detectors, with and without the normal noise of s%

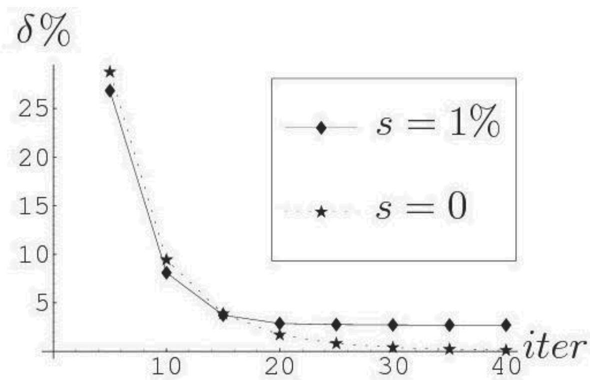


Fig. 18. Dependence of the error $\delta\%$ on the number of iterations in simulation of the algorithm CHRB-3 for resolution 20x20, 18 sources and detectors, with and without the normal noise of s%

5. Conclusions

In this paper we have presented the basic algorithms of computer tomography which can be applied in the standard approach or, after the proper adaptation, in solving the problems of the incomplete projection data (arising in certain technical issues mentioned in the previous sections). We have considered the general model of asynchronous iterations and new block, chaotic and chaotic-block iterative algorithms for reconstruction of the high-contrast objects from incomplete projection data. These algorithms can be realized on the parallel computing structure consisting of elementary processors and some central processor, all of which are connected with shared memory. We have studied the quality and convergence of these algorithms by computing simulation on sequential computer. The experimental results show that convergence of the block-parallel chaotic algorithm CHRB-3 is better by comparison with the block-parallel algorithm RB-3, the iterative-block algorithm SZB-3 and especially with standard algorithms.

Taking into account the time of implementation, for the block-parallel algorithm implemented on multiprocessors computers the time is approximately *M* times smaller (where *M* is the number of processors which is equivalent to the number of blocks) than for the sequential computers. Additionally, from results of computer simulation it follows that the time of running of the block-parallel algorithms is better with comparison with the sequential ART-3.

From our results it also follows that the configuration (1×1, 1×1) is considerably better by comparison with the scheme (1×1). And for each considered scheme of reconstruction there exist the parameters which allow to obtain the enough good quality of reconstruction after some number of iterations but this number is considerably larger than for reconstruction with complete projection data. However, in some technical problems (like for example in examining the coal bed where the access to the researched object is significantly limited - see the Figure 1) it is impossible to use the scheme (1×1, f1×1). The received results indicate that the discussed algorithms are convergent also for the system (1×1), but for receiving the same reconstruction quality they need the considerably bigger number of iterations. Moreover, all the considered algorithms are stable.

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