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Computer modelling and analysis of microstructures with fibres and cracks

P. Fedeliński *
Department of Strength of Materials and Computational Mechanics,
Silesian University of Technology, ul. Konarskiego 18a, 44-100 Gliwice, Poland
* Corresponding e-mail address: piotr.fedelinski@polsl.pl

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Analysis and modelling

ABSTRACT

Purpose: The aim of the research is to formulate the boundary element approach, develop the computer codes and analyze microstructures containing fibres and cracks. The computer codes can be used to analyze influence of fibres and cracks on stress fields and effective properties of materials.

Design/methodology/approach: The relation between boundary displacements and tractions is established by using appropriate boundary integral equations. The variations of boundary coordinates, displacements and tractions are interpolated by using nodal values and shape functions. Additionally, equations of motion and equilibrium equations are applied for rigid fibres.

Findings: The boundary element method can be simply and effectively used for materials containing fibres and cracks. The stress fields for a single fibre computed by the present approach agree very well with analytical results. The fibre, which is perpendicular to the crack has larger influence on stress intensity factors than the fibre, which is parallel to the crack.

Research limitations/implications: The proposed method is efficient for linear elastic materials. For other materials the boundary element method requires fundamental solutions, which have complicated forms. The developed computer codes can be extended to materials containing many randomly distributed fibres and cracks.

Practical implications: The present method can be used to analyze and optimize strength and stiffness of materials by a proper reinforcement by fibers.

Originality/value: The original value of the paper is the analysis of influence of distribution of rigid fibres on effective properties of composites and the influence of positions of a fibre and a crack on stress intensity factors.

Keywords: Computational mechanics; Boundary element method; Fibre; Crack; Effective properties

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1. Introduction

Fibres are used in composite materials in order to increase strength and stiffness of structures. If the stiffness of fibers is much greater than the matrix then the fibers can be modelled as rigid stiffeners in a deformable matrix. An overview of methods for analysis of solids with inclusions and cracks was presented by Mura [1].

Single rigid fibres in an infinite domains were analyzed by analytical methods and composites containing many fibres - by computational methods. Dundurs and Markenscoff [2] presented a Green's function formulation for rigid thin lamellar inclusions that are bounded to the surrounding material. The authors call such inclusions anticracks because they transmit tractions and prevent displacement discontinuity. The paper presents stress fields for an anticrack loaded by concentrated forces, a concentrated couple and an edge dislocation. Li and Ting [3]

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analyzed a line inclusion in an anisotropic elastic infinite plate subjected to uniform loading at infinity. They used the Stroh formalism to calculate the displacement and stress fields for the rigid and elastic inclusion. Liu et al. [4] applied the fast multipole boundary element method (BEM) to analyze composites reinforced by nanotubes. They calculated effective material properties of composites modelled as three-dimensional structures having large number of degrees of freedom.

Pingle et al. [5] used the duality principle to derive rigid line inclusion solutions from the crack solutions. They computed stress fields around a rigid line inclusion and derived a compliance contribution tensor for a single and multiple line inclusions. Gorbatikh et al. [6] determined the relation between stress intensity factors at the tips of rigid line inclusions and effective compliance of the material.

A problem of interaction between rigid-line inclusions and cracks was presented in several papers. Hu et al. [7] analyzed an interaction of cracks and rigid fibers near the interface between different materials using integral equations. Salgado and Aliabadi [8] modelled plates with cracks reinforced by stiffeners by the dual BEM. Plates with many growing cracks were considered. Dong et al. [9] used the BEM to analyze an interaction of cracks and fibers in infinite plates. Dong and Lee [10] used the boundary integral approach to analyze double periodic array of cracks/rigid-line inclusions in an infinite isotropic plate. The method was used to compute stress intensity factors and effective elastic properties. Dong [11] presented the integral equation formulation for infinite homogenous isotropic plate containing inclusions, rigid lines and cracks. The influence of distance and material properties of inclusions on stress intensity factors were studied.

The boundary element method (BEM) is a versatile computer method which is used in different areas of solid mechanics [12]. One of the new areas of the BEM is mechanics solids containing many cracks and inclusions. Such problems can be efficiently analyzed by the BEM, which requires discretization of the external boundary, fibres and crack edges. The formulation and application of the BEM for a single rigid fibre in a finite plate was presented by Fedelinski [13]. In the present work the method is extended and applied to solids with multiple rigid-line reinforcements. The analysis of representative volume elements containing many randomly distributed cracks was shown by Fedelinski [14]. The influence of the volume density of cracks on stress intensity factors and overall stiffness was presented.

The aim of this work is to present the formulation and numerical results for statically loaded, linear elastic and isotropic finite plates with rigid straight fibers and cracks obtained by the BEM. The paper shows the influence of distribution of rigid fibres on effective properties of composites and an influence of positions of a fibre and a crack on stress intensity factors.

2. Boundary element method for plates with rigid fibres

2.1. Boundary integral equations for a plate with rigid fibres

Consider a plate made of homogenous, isotropic and linear elastic material. The boundary of the plate is denoted by Γ and its domain by Ω (Fig. 1). The plate is statically loaded along the external boundary Γ by boundary tractions t_j and the domain Ω by body forces f_j. The relation between the loading of the plate and its displacements can be expressed by the Somigliana identity [12]

$$c_{ij}(x')u_j(x') + \int_{\Gamma} T_{ij}(x',x)u_j(x)d\Gamma(x) = \int_{\Gamma} U_{ij}(x',x)t_j(x)d\Gamma(x) + \int_{\Omega} U_{ij}(x',X)f_j(X)d\Omega(X) \quad , \tag{1}$$

where: x' is the collocation point, for which the above integral equation is applied, x is the boundary point, X is the domain point and c_{ij} is a constant, which depends on the position of the point x', U_{ij} and T_{ij} are the Kelvin fundamental solutions of elastostatics. In equations the Einstein summation convention is used and the indices for two-dimensional problems have values $i,j=1,2$.

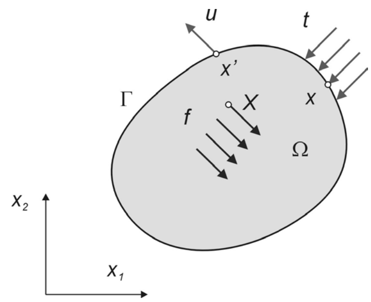


Fig. 1. Elastic plate - loading and displacements

Assume that the plate is reinforced by straight, thin and rigid fibres, which are perfectly bonded to the matrix. If the plate is deformed then forces of interaction occur along the lines of attachment of fibers (Fig. 2). These forces of interaction can be treated as particular body forces acting along the lines in the domain of the body. The boundary integral equation (1) for the plate loaded by boundary tractions and forces of interaction of fibres has the form

$$c_{ij}(x')u_j(x') + \int_{\Gamma} T_{ij}(x',x)u_j(x)d\Gamma(x) = \int_{\Gamma} U_{ij}(x',x)t_j(x)d\Gamma(x) + \sum_{n=1}^{N} \int_{\Gamma_n} U_{ij}(x',X)t_j^n(X)d\Gamma_n(X) \quad , \tag{2}$$

where: N is the number of fibres, Γ_n is the line of attachment and t_j^n are the forces of interaction.

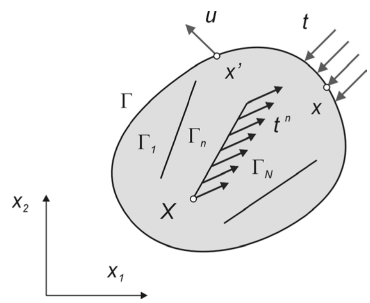


Fig. 2. Elastic plate with fibres

2.2. Displacements and equations of equilibrium for rigid fibres

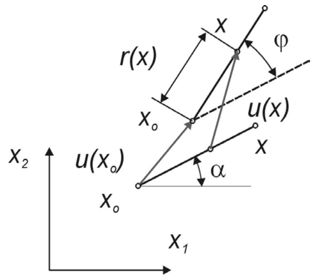


Fig. 3. Displacements of the rigid fibre

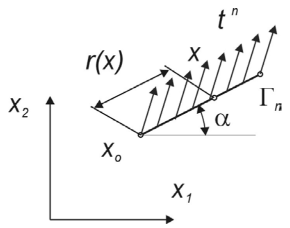


Fig. 4. Forces acting on the fibre

Deformations of the matrix influence displacements of fibres. The displacement of an arbitrary point x of the fibre can be expressed by the displacements of the fibre tip x_o and the angle of rotation of the fibre φ (Fig. 3). For small angles of rotation of fibres, the components of displacements of an arbitrary point of the fibre are expressed in the form

$$u_1(x) = u_1(x_o) - \varphi r(x)\sin\alpha \quad , \tag{3}$$

$$u_2(x) = u_2(x_o) + \varphi r(x)\cos\alpha \quad , \tag{4}$$

where: α is the initial angle between the fibre and the axis x_1 of the global coordinate system, r is the distance between the point x and the fibre tip x_o.

The considered structure and therefore each fibre is in equilibrium. The forces acting on each fibre (Fig. 4) should satisfied the following equilibrium equations

$$\int_{\Gamma_n} t_1^n(x)d\Gamma_n(x) = 0 \quad , \tag{5}$$

$$\int_{\Gamma_n} t_2^n(x)d\Gamma_n(x) = 0 \quad , \tag{6}$$

$$\int_{\Gamma_n} [-t_1^n(x)\sin\alpha + t_2^n(x)\cos\alpha]r(x)d\Gamma_n(x) = 0 \quad . \tag{7}$$

The last equation (7) is the equation of moments of forces with respect of the fibre tip x_o.

2.3. Numerical implementation of the method

The boundary of the plate and the fibres are divided into boundary elements (Fig. 5). In the developed computer code 3-node quadratic boundary elements are used. Along the external boundary of the plate the variations of coordinates, displacements and tractions, and along the fibres the variations of interaction forces are interpolated. The boundary integral equations (2) are used for nodes along the external boundary and the fibres.

The displacements of fibre nodes can be expressed by the displacements of fibre tips and their angles of rotation, by using Eqs (3) and (4). These equations can be written in the following matrix form

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u}_f \quad , \tag{8}$$

where the matrix $\boldsymbol{u}$ contains the components of displacements of fibre nodes, the matrix $\boldsymbol{I}$ depends on the position of nodes and the matrix $\boldsymbol{u}_f$ contains components of displacements of fibre tips and their angles of rotation.

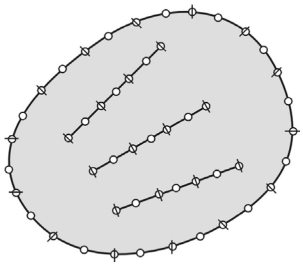


Fig. 5. Discretization of the matrix and fibres by quadratic boundary elements

The equilibrium equations for fibres (5), (6) and (7) can be written in the matrix form

$$\boldsymbol{E}\boldsymbol{t}_f = 0 \quad , \tag{9}$$

where the matrix $\boldsymbol{E}$ depends on the position of fibre nodes and the matrix $\boldsymbol{t}_f$ contains nodal values of components of tractions in fibres. The matrix $\boldsymbol{E}$ is obtained by integration of expressions in Eqs (5), (6) i (7), by assuming quadratic variations of forces along the fibres. Because the equilibrium equations have very simple forms, the integrals are computed analytically.

Boundary integral equations (2), supplied with Eqs (8) and (9) can be written in the matrix form

$$\begin{bmatrix} \boldsymbol{H}_{ee} & 0 \\ \boldsymbol{H}_{fe} & \boldsymbol{I} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{u}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{ee} & \boldsymbol{G}_{ef} \\ \boldsymbol{G}_{fe} & \boldsymbol{G}_{ff} \\ 0 & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_e \\ \boldsymbol{t}_f \end{bmatrix} \quad , \tag{10}$$

where the submatrices with the index $\boldsymbol{e}$ are related to the external boundary and the submatrices denoted by the index $\boldsymbol{f}$ are related to the fibres. The submatrices $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on boundary integrals of fundamental solutions, shape functions and are integrated numerically by using the Gauss quadrature.

Next the system of algebraic equations is rearranged. The unknown quantities are on the left side of the equation and the known quantities on the right side of the equation. The first modification refers to the unknown interaction forces $\boldsymbol{t_f}$

$$\begin{bmatrix} \boldsymbol{H}_{ee} & -\boldsymbol{G}_{ef} & 0 \\ \boldsymbol{H}_{fe} & -\boldsymbol{G}_{ff} & \boldsymbol{I} \\ 0 & \boldsymbol{E} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{t}_f \\ \boldsymbol{u}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{ee} \\ \boldsymbol{G}_{fe} \\ 0 \end{bmatrix} [\boldsymbol{t}_e] \quad . \tag{11}$$

In the final modification the known and unknown boundary conditions are rearranged. The modified matrix equation is solved and the unknown displacements and tractions along the external boundary and the displacements and interaction forces for fibres are obtained.

3. Boundary element method for plates with cracks

3.1. Boundary integral equations for a plate with cracks

The method is applied to a linear elastic, homogeneous and isotropic body containing a crack. The boundary of the body Γ consists of the external boundary Γ^e and two crack surfaces Γ^+ and Γ^-, as shown in Fig. 6.

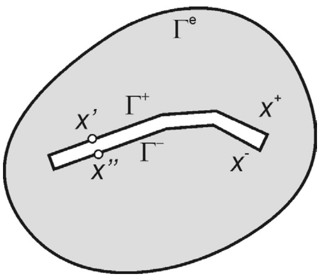


Fig. 6. Elastic plate with a crack

The displacement equation for points, which belong to the smooth crack surfaces has the form [15]:

$$\frac{1}{2}u_i(x') + \frac{1}{2}u_i(x'') = \int_\Gamma U_{ij}(x',x)t_j(x)d\Gamma(x) - \int_\Gamma T_{ij}(x',x)u_j(x)d\Gamma(x) \quad , \tag{12}$$

where x' and x'' are coincident points on the opposite crack surfaces. The traction equation for the same points is [15]:

$$\frac{1}{2}t_j(x') - \frac{1}{2}t_j(x'') = n_i(x')\{\int_\Gamma U_{kij}(x',x)t_k(x)d\Gamma(x)] - \int_\Gamma T_{kij}(x',x)u_k(x)d\Gamma(x)\} \quad , \tag{13}$$

where $U_{kij}(x',x)$ and $T_{kij}(x',x)$ are the fundamental solutions of elastostatics and $n_i(x')$ is an outward normal unit vector at the collocation point.

3.2. Numerical implementation of the method

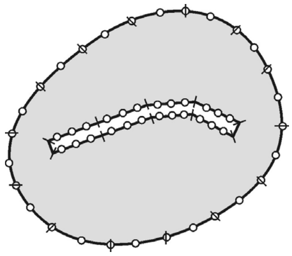


Fig. 7. Discretization of the plate with a crack by quadratic boundary elements

The boundary Γ of the body is divided into boundary elements. A distinct set of boundary integral equations is obtained by applying the displacement equation (1) for collocation nodes along the external boundary, the displacement equation (12) and the traction equation (13) simultaneously for coincident nodes along both crack faces. Quadratic elements are used for the discretization of the boundary, as shown in Fig. 7. After discretization and integration the following matrix equation is obtained:

$$\begin{bmatrix} \boldsymbol{H}^{ee} & \boldsymbol{H}^{e+} & \boldsymbol{H}^{e-} \\ \boldsymbol{H}^{+e} & \boldsymbol{H}^{++} & \boldsymbol{H}^{+-} \\ \boldsymbol{H}^{-e} & \boldsymbol{H}^{-+} & \boldsymbol{H}^{--} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^e \\ \boldsymbol{u}^+ \\ \boldsymbol{u}^- \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}^{ee} & \boldsymbol{G}^{e+} & \boldsymbol{G}^{e-} \\ \boldsymbol{G}^{+e} & \boldsymbol{G}^{++} & \boldsymbol{G}^{+-} \\ \boldsymbol{G}^{-e} & \boldsymbol{G}^{-+} & \boldsymbol{G}^{--} \end{bmatrix} \begin{bmatrix} \boldsymbol{t}^e \\ \boldsymbol{t}^+ \\ \boldsymbol{t}^- \end{bmatrix} , \tag{14}$$

where $\boldsymbol{u}^e$, $\boldsymbol{u}^+$, $\boldsymbol{u}^-$ and $\boldsymbol{t}^e$, $\boldsymbol{t}^+$, $\boldsymbol{t}^-$ contain nodal values of displacements and tractions along the external and opposite crack surface; the submatrices $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on the fundamental solutions and interpolating functions. The columns of matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are reordered according to the boundary conditions.

The matrix equation is solved giving the unknown displacements and tractions along boundaries of the body. The stress intensity factors (SIF) are computed by using the path independent integral [15].

4. Numerical examples

To demonstrate the accuracy of the method and possible applications several numerical examples are solved.

4.1. Single rigid fibre in an infinite plate

An infinite plate with a rigid fibre of length $2l$ is subjected to the parallel loading q_1 or to the perpendicular loading q_2, as shown in Fig. 8. The structure is modelled as a finite square plate with a rigid fibre and the dimensions of the plate are 10 times larger than the fibre. The material of the plate has the Poisson ratio ν=0.25 and is in plane stress state. The rigid fibre is divided into 20 boundary elements and the square plate into 160 boundary elements. Stresses are computed at 245 internal points in the neighbourhood of the fibre. This field is marked as a grey square in Fig. 8.

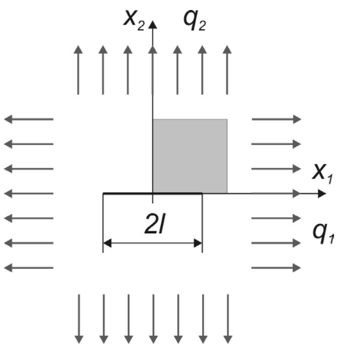


Fig. 8. Rigid fibre in an infinite plate - dimension and loading

a)

b)

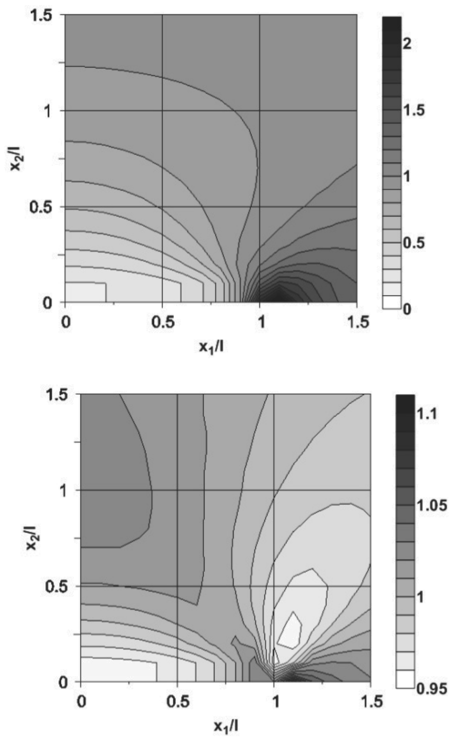


Fig. 9. Normal stresses in the vicinity of the rigid fibre: a) parallel loading - normalized stress σ_{11}/q_1; b) perpendicular loading - normalized stress σ_{22}/q_2

The contour plot of normalized stresses σ_{11}/q_1 for the parallel loading q_1 and normalized stresses σ_{22}/q_2 for the perpendicular loading q_2 are shown in Fig. 9.

For the parallel loading, stresses σ_{11} have the smallest values along the fibre and the largest values along the extension of the fibre. For the perpendicular loading the stresses σ_{22} are more uniformly distributed in the analyzed field. The maximum values of stresses exceed the applied tension by about 2%. The distribution of stresses agree very well with stresses computed analytically and presented by Pingle et al. [5].

4.2. Multiple rigid fibres in a rectangular plate

A rectangular plate of length $2b$ and height $2h$ contains 13 rigid fibres of length $2l$, as shown in Fig. 10. The horizontal distance between centres of neighbouring fibres is d_1 and the vertical distance is d_2. The ratios of dimensions are: b/l=5, h/l=4, d_1/l=3 and d_2/l=1.6. The material of the plate has the Poisson ratio ν=0.3 and is in plane strain state. The plate is subjected to the horizontal loading q_1. Each rigid fibre is divided into 8 boundary elements and the external boundary into 72 boundary elements.

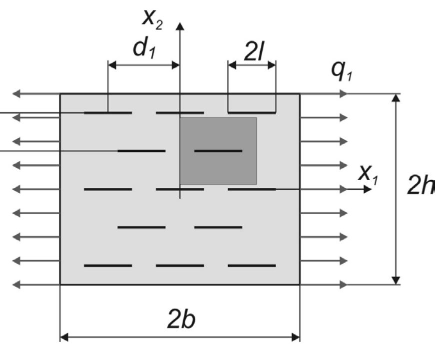


Fig. 10. Rigid fibres in a rectangular plate - dimensions and loading

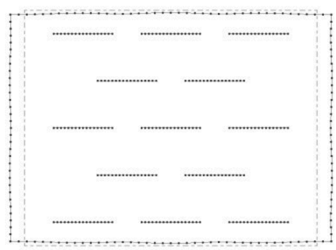


Fig. 11. Rigid fibres in a rectangular plate - initial shape (dashed line) and deformed shape (continuous line)

The initial shape and the deformed shape of the plate are shown in Fig. 11. The stress distributions in the matrix, in the marked field in Fig. 10, are shown in Fig. 12. The relative effective Young modulus is computed as E_c/E_m=1.345, where E_c and E_m are the Young modulus of the composite and the matrix, respectively. The effective Poisson ratio of the composite is the same as for the matrix.

The influence of the horizontal and vertical distance between the fibres on the effective Young modulus is analysed. The dimensions of fibres are constant and the dimensions of the plate are changed proportionally to the distance between the fibres. The dependence of the relative Young modulus on the horizontal and vertical distance is presented in Fig. 13. If the horizontal or vertical distance are smaller than half of the length of the fibre the stiffness of the composite significantly increases. For the increasing distance between the fibres the stiffness of the composite tends to the stiffness of the matrix.

a)

b)

c)

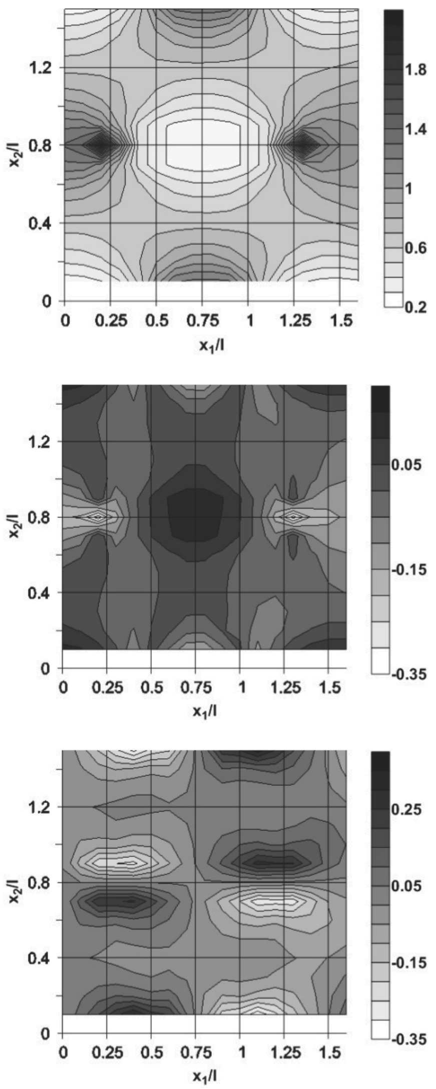


Fig. 12. Stress distributions in the matrix: a) σ_{11}/q_1, b) σ_{22}/q_1, c) σ_{12}/q_1

4.3. Interaction of a fibre with a crack

A square plate contains a crack of length $2a$ in the centre and a fibre of length $2l$, as shown in Fig. 14. The dimensions of the plate $b=h$ are 10 times larger than the dimension of the crack a. Therefore it can be assumed that the plate has infinite dimensions. The distance between the fibre and the crack is d. The influence of the direction of the fibre (parallel and perpendicular, as shown in Fig. 15) with respect to the crack and different distances d on stress intensity factors (SIF) is studied. The material properties of the plate are: the Young modulus $E=2\cdot 10^{11}$ Pa, the Poisson ratio $\nu=0.3$ and the plate is in plain strain state.

a)

b)

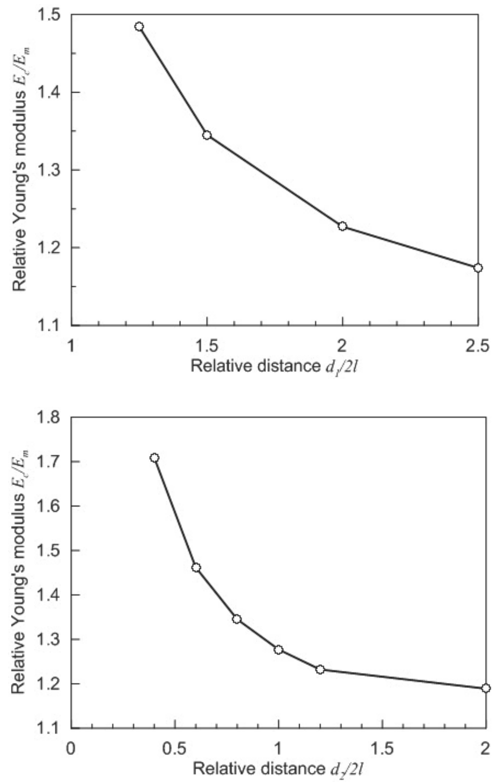


Fig. 13. Influence of the distance between the fibres on the relative effective Young modulus E_c/E_m: a) horizontal distance $d_1/2l$, b) vertical distance $d_2/2l$

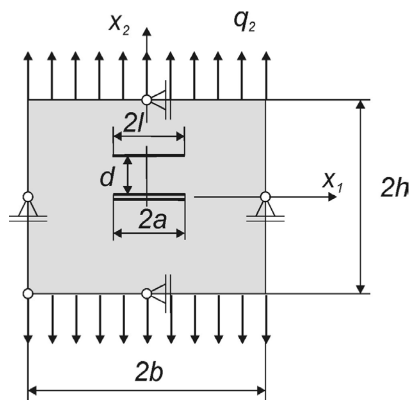


Fig. 14. Square plate with a crack and a fibre

a) b)

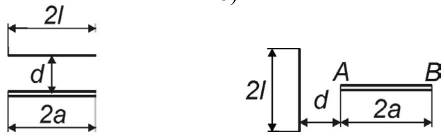


Fig. 15. Fibre: a) parallel, b) perpendicular to the crack

The plate is loaded in the vertical direction by the uniformly distributed tractions q_2 and supported along the lines of symmetry. The external boundary of the plate is divided into 160 elements, the crack edges - into 40 elements and the fibre - into 20 quadratic boundary elements. The stress intensity factors are normalized with respect to the SIF for the crack in the infinite plate $K_o = q_2\sqrt{\pi a}$. The normalized SIF for the crack in the considered square plate without the fibre is K_I/K_o=1.016.

The influence of the parallel fibre to the crack on SIFs is studied. Fig 16 shows the influence of the relative distance d/a for the fibre, which has the same length as the crack. For small distance, the fibre increases SIFs, however the influence is small.

a)

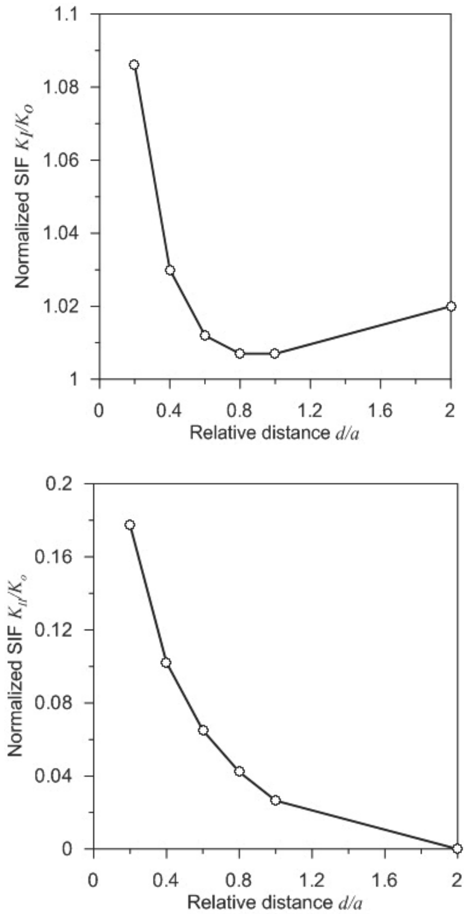


Fig. 16. Influence of the relative distance d/a on the normalized SIF: a) K_I/K_o, b) K_{II}/K_o

Fig. 17 shows the influence of the relative length of the fibre l/a on SIFs for the relative distance d/a=0.4. For the fibre smaller than the crack the fibre has small influence on SIFs. For longer fibres the K_I/K_o decreases and K_{II}/K_o increases.

a)

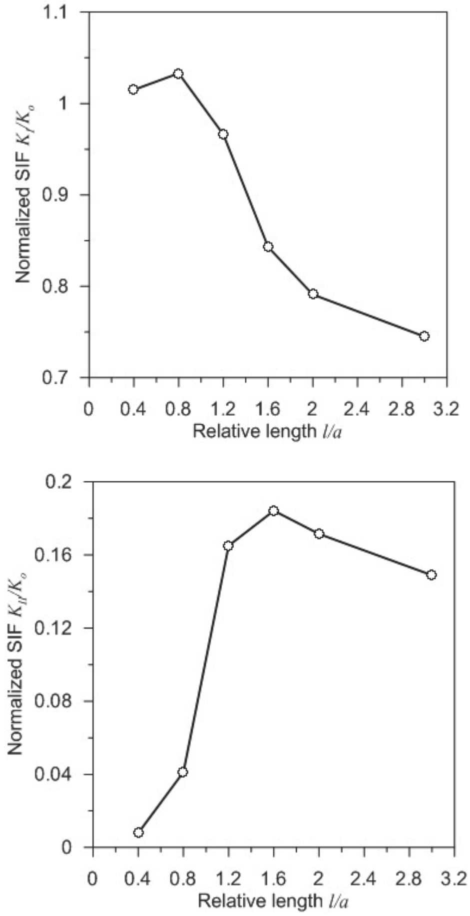


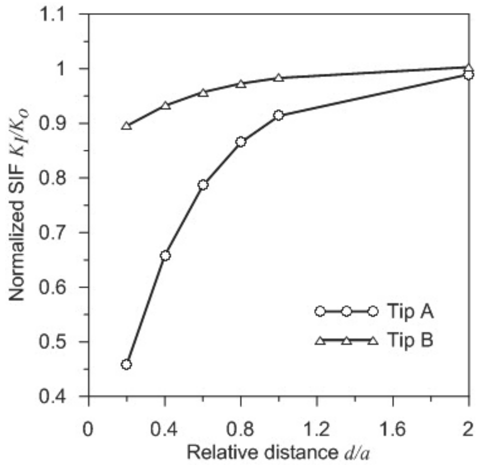
Fig. 17. Influence of the relative fibre length l/a on the normalized SIF: a) K_I/K_o, b) K_{II}/K_o

Next, the influence of the perpendicular fibre to the crack on the SIF is studied. Fig. 18a shows the influence of the relative distance d/a for the fibre having the same length as the crack and Fig. 18b - the influence of the relative length l/a for constant distance d/a=0.4. The perpendicular fibre reduces significantly SIFs in comparison to the parallel fibre. As expected, the influence on the crack tip which is closer to the fibre is stronger.

5. Conclusions

The boundary element method is a very efficient method for modelling elastic bodies containing rigid fibres and cracks. The modelling of such materials is significantly simplified in comparison to domain methods, for example the finite element method, because nodes are situated only along the external boundary, fibers and crack edges. Positions, shapes and dimensions of fibers and cracks and their numbers can be very simply modified.

a)



b)

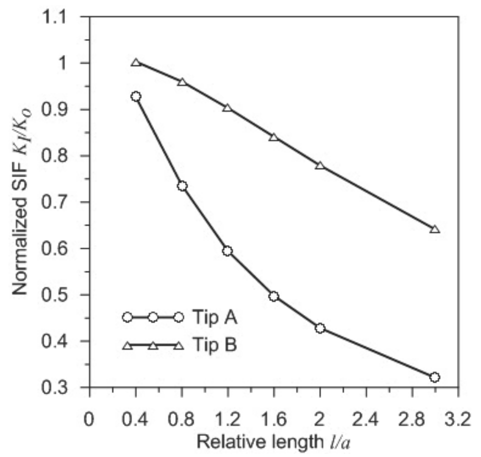


Fig. 18. Influence of: a) relative distance d/a, b) relative length l/a, on the normalized SIF K_I/K_o at the crack tips A and B

The comparison of stresses for the fibre in the infinite plate obtained by the present method and the analytical method shows that the proposed approach gives very accurate results. For distances between the fibres smaller than the half length of fibres an influence of interaction of fibres on stress distribution in the matrix and significant increase of stiffness can be observed.

The fibre, which is perpendicular to the crack has larger influence on stress intensity factors than the fibre, which is parallel to the crack. The perpendicular fibre, which is close to the crack tip, can significantly reduce stress intensity factors.

Acknowledgements

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