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Study of generalized Prandtl rheological model for constitutive description of elastoplastic properties of materials

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Analysis and modelling

ABSTRACT

Purpose: of this paper is to demonstrate the process of constitutive modelling of elastoplastic properties of materials using generalized Prandtl rheological model. A special attention is put on description of composites.
Design/methodology/approach: Based on the proposed rheological model, the set of constitutive relationships is formulated. Identification of parameters of rheological model is carried out based on experimental hysteretic loops. The constitutive equations are used in the paper for computer simulation of experimental tests.
Findings: It is proved in the paper that the obtained constitutive relationships can describe the phenomenon of plastic anisotropy. An illustrating example is demonstrated for fiber glass-reinforced polymer-matrix composite. The comparison between experimental results and computer simulations shows the validity of the model.
Research limitations/implications: The computer simulations concentrate on one-dimensional problem. It is suggested for future investigations to implement three-dimensional constitutive model. Such an implementation may be conducted within FEM codes ABAQUS or ANSYS.
Practical implications: Using the method of constitutive modelling of elastoplastic properties of materials it is possible to carry out computer simulations solving non-linear differential equations for any type of loadings both static and dynamic.
Originality/value: The original value of the paper is the proposed procedure of identification of material model exhibiting plastic anisotropy based on generalized Prandtl rheological scheme. As the result, the system of constitutive relationships has explicit differential form, easy for numerical implementations.
Keywords: Generalized Prandtl model; Constitutive relationships; Elastoplasticity; Plastic anisotropy; Rheological schemes; Hysteresis; Composites

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1. Introduction

Composite materials are commonly used for construction, engineering and other applications [1-3]. They are formed using no less than two materials being still distinguishable and not fully blended. Composite should have better properties in comparison with the properties of its components. Composites take advantage of the strengths and abilities of the components.

Analyzing the stress and strain states of composite materials needs appropriate constitutive models to be taken into consideration. Such models should describe elastoplastic properties of materials [4].

The purpose of this paper is to demonstrate the procedure of constitutive modelling for materials, including composites, exhibiting elastoplastic properties. The procedure is based on the analysis of generalized Prandtl rheological model. The obtained set of constitutive equations is defined by parameters of the rheological scheme. Identification of these parameters is carried out based on experimental hysteretic loops. Moreover, the model can describe the phenomenon of plastic anisotropy caused by different strengths of materials in tension and compression.

The constitutive relationships proposed in the paper have a form of non-linear explicit differential equations. The implementation of these equations is demonstrated via an example in which the experimental hysteretic loop for fiber glass-reinforced polymer-matrix composite is compared with results of computer simulations.

The procedure of constitutive modelling of materials using rheological schemes was previously applied by authors for metallic alloys, shape memory alloys (SMA) and asphalt-aggregate mixes [5-8].

2. Stress and strain state description

The stress and the strain state of a body is described by the symmetrical 2nd order tensors. Let $\mathbb{S}$ and $\mathbb{E}$ be the Euclidean linear spaces, for stress and strain respectively. The elements belonging to the above spaces are represented as 3×3 matrixes. The inner product of two 2nd order tensors is denoted $\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\in\mathbb{R}^1$ where $\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)}\in\mathbb{S}$, and can be defined as follows

$$\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}=\sigma_{ij}^{(1)}\sigma_{ij}^{(2)};\quad i,j=1,2,3 \tag{1}$$

The summation convention is assumed over the repeated indices. The norm of a tensor is denoted $\|\boldsymbol{\sigma}\|\in\mathbb{R}^1$ and is defined by

$$\|\boldsymbol{\sigma}\|:=\sqrt{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}=\sqrt{\sigma_{ij}\sigma_{ij}} \tag{2}$$

The space of the stress and strain tensors may be decomposed into two orthogonal sub-spaces. The first sub-space constitutes the sub-space of spherical tensors. The second one is the sub-space of deviatoric tensors. Let us assume the following symbols to be used for the stress and strain state description

$$\boldsymbol{\sigma}=\mathbf{p}+\mathbf{s};\ \boldsymbol{\varepsilon}=\mathbf{a}+\mathbf{e}\ \text{ where }\ \mathbf{p}=\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}\ \text{ and }\ \mathbf{a}=\frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \tag{3}$$

Here **p** and **a** denote spherical tensors, **s** and **e** are deviators and **I** denotes identity 2nd order tensor. In the above equations the operation $\mathrm{tr}\,\boldsymbol{\sigma}=\sigma_{ii}\in\mathbb{R}^1$ denoting the trace of a tensor was used additionally.

3. Elastoplastic properties of materials

Mathematical description of elastic properties of isotropic bodies can be formulated separately for the spherical and deviatoric sub-spaces. The Hooke's law may be written in the form of two linear equations

$$\mathbf{p}=3K\mathbf{a}\,;\quad \mathbf{s}=2G\mathbf{e} \tag{4}$$

where K and G denote bulk modulus and shear modulus respectively.

Analyzing the constitutive properties of elastoplastic material, the hypothesis stating that the deviatoric part of the strain tensor is decomposed into two components may be assumed

$$\mathbf{e}=\mathbf{e}_{\mathrm{el}}+\mathbf{e}_{\mathrm{pl}} \tag{5}$$

where the deviators $\mathbf{e}_{\mathrm{el}}$ and $\mathbf{e}_{\mathrm{pl}}$ are related to the elastic and the plastic part of the strain state.

The stress state in elastic perfectly plastic material is described by two relations

$$\mathbf{p}=3K\mathbf{a}\,;\quad \mathbf{s}=2G(\mathbf{e}-\mathbf{e}_{\mathrm{o}}) \tag{6}$$

The constitutive properties of the plastic pressure-independent material may be described in the form of the following inclusion and variational inequality [9,10]

$$\begin{gathered}\mathbf{s}\in\Theta\\ \dot{\mathbf{e}}_{\mathrm{o}}\cdot(\mathbf{s}-\tilde{\mathbf{s}})\geq 0\quad\forall\,\tilde{\mathbf{s}}\in\Theta\end{gathered} \tag{7}$$

where the set Θ determines admissible stresses in deviatoric sub-space and the superposed dot denotes differentiation with respect to the time coordinate. The relationship $(6)_2$ is well-known as Drucker's stability postulate or Hill's principle of maximum plastic work.

Taking into account the Huber-Mises-Hencky's (HMH) yield criterion the set Θ is described as

$$\Theta:=\left\{\mathbf{s}\in\mathbb{S}:\ \ \|\mathbf{s}\|\leq\sqrt{2}\,k\right\} \tag{8}$$

Here k denotes the value of yield limit obtained via pure shear test. The relationship between k and the tension/compression yield limit σ_o in case of HMH criterion is as follows $\sigma_o = \sqrt{3}\,k$.

As a consequence of introduction Eqn. (8), the relations (7) may be rewritten in the following form

$$\dot{\mathbf{e}}_{pl} = \lambda \mathbf{s}$$
$$\|\mathbf{s}\|^2 \le 2k^2; \quad \lambda \ge 0; \quad \lambda\left(\|\mathbf{s}\|^2 - 2k^2\right) = 0 \tag{9}$$

where the Eqn. $(9)_1$ is well-known associated flow rule while the relations $(9)_2$ are loading/unloading or Kuhn-Tucker conditions. The scalar λ denotes so called Lagrange multiplier. The procedure of evaluation of Lagrange multiplier for elastic-perfectly plastic material leads to the following equation describing plastic deviatoric strain rate [11]

$$\dot{\mathbf{e}}_{pl} = \begin{cases} \mathbf{0} & \text{if} \quad \|\mathbf{s}\| < \sqrt{2}\,k \\ \dfrac{\mathbf{s}}{2k^2}\left[\mathbf{s}\cdot\dot{\mathbf{e}}\right]^+ & \text{if} \quad \|\mathbf{s}\| = \sqrt{2}\,k \end{cases} \tag{10}$$

where the function $[.]^+$ denotes projection onto the set of non-negative numbers being defined as follows

$$[z]^+ := \begin{cases} z & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The set of constitutive Equations (5), (6) and (10) defines simple Prandtl elastoplastic model. In case of deviatoric stresses the Prandtl model can be represented by rheological scheme shown in Fig. 1. The spring element represents elastic properties while the slider represents rigid perfectly plastic model.

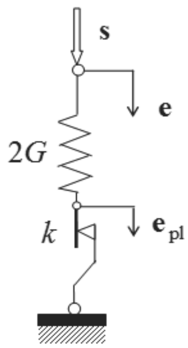

Fig. 1. Rheological scheme of simple Prandtl model in deviatoric subspace

Let us move back to the Eqn. (8) defining the set of admissible stresses corresponding to the HMH criterion. Because of the fact that lots of materials exhibit different limit stresses in tension and compression, the non-symmetric HMH criterion may be used [12]. Let us assume the following relation between tension yield limit σ_o^+ and compression yield limit σ_o^-

$$\sigma_o^- = \alpha\,\sigma_o^+ \text{ where } \alpha \in \langle 0,1 \rangle \tag{12}$$

Taking into account Eqn. (12) the set Θ can be described as follows

$$\Theta := \left\{ \mathbf{s} \in \mathbb{S}: \ \|\mathbf{s} - \mathbf{s}_o\| \le \sqrt{2}\,k \right\} \tag{13a}$$

where the tensor $\mathbf{s_o}$ determines translation of the yield surface in stress space. A possible form of such a tensor is

$$\mathbf{s}_o = \frac{1-\alpha}{2}\sigma_o^+ \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \tag{13b}$$

Let us note that analyzing Eqn. 13b, for $\alpha = 1$, then $\sigma_o^+ = \sigma_o^- = \sigma_o$ and the set Θ described by Eqn. (13a) transforms to the symmetrical HMH condition given in Eqn. (8).

4. Hysteretic loop

Elastic-dissipative properties of deformable bodies can be characterized by an operator F, mapping the strain time history function $\boldsymbol{\varepsilon}(t)$ into the stress function $\boldsymbol{\sigma}(t)$

$$\boldsymbol{\sigma}(t) = \mathsf{F}(\boldsymbol{\varepsilon})(t) \tag{14}$$

In case of a body possessing linear properties, the operator F has a form of integral Boltzman operator. In this paper the form of the operator F will be determined based on mathematical description of rheological schemes modelling constitutive properties of materials. In order to characterize the stress and the strain states, scalar quantities are used very often. In such a case, based on Eqn. (14) we can create the graph of a parametric function illustrating the relationship between analyzed scalar quantities. The curve obtained in such a way is called hysteretic loop [13].

Hysteretic loops for deformable bodies are used to be created for a cyclic, one-dimensional loading of a material. Thus, the hysteretic loop can be visualized in the following planes: sample elongation u versus axial load f or axial strain ε versus axial stress σ. It is worth mentioning that hysteretic loop for materials possessing linear properties has the shape of ellipse.

For further investigations we will analyze hysteretic loops determined for one-dimensional tests and visualized on the plane (ε,σ) or (u,f). Such hysteretic loops are used very often in the process of identification of rheological models.

As an example let us analyze one-dimensional simple Prandtl model. The three-dimensional relationships as well as the rheological scheme were presented in previous chapter. Figure 2 visualizes one-dimensional simple Prandtl model and its hysteretic loop.

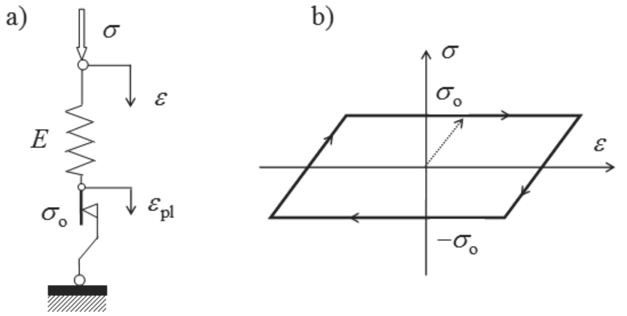

Fig. 2. Rheological scheme (a) and hysteretic loop (b) of one-dimensional simple Prandtl model

The spring element shown in Fig. 2a is characterized by Young modulus E obtained via tension/ compression test. One-dimensional plastic property of the model, represented by the slider, is shown in Fig. 3. Two cases were considered. Figure 3a shows symmetrical tension/compression model while in Fig. 3b the non-symmetrical model is shown.

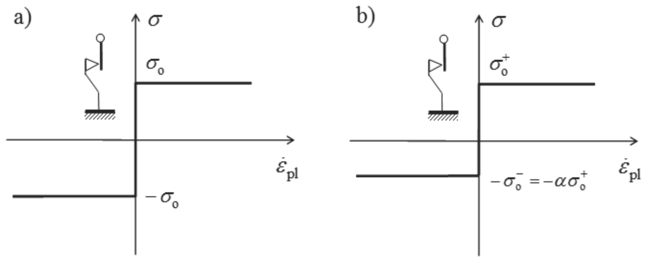

Fig. 3. One-dimensional plastic characteristics for symmetrical (a) and non-symmetrical (b) tension/compression models

Another example of simple hysteretic loop concerns one-dimensional elastoplastic model with hardening (simple Prandtl model with hardening). The scheme is shown in Fig. 4a while the hysteretic loop in Fig. 4b. As we can see, in order to model linear hardening phenomenon, simple modification of rheological scheme was done (compare Figs. 2a and 4a).

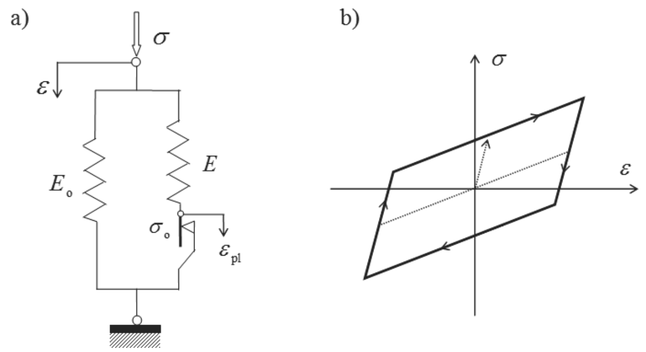

Fig. 4. Rheological scheme (a) and hysteretic loop (b) of one-dimensional simple Prandtl model with hardening

5. Generalized Prandtl model

A rheological scheme of generalized Prandtl model with hardening is shown in Fig. 5. This structure contains $N \to \infty$ Prandtl networks and one spring in parallel. The hysteretic loop of generalized Prandtl model is shown in Fig. 6. The loop can be approximated by two curves φ and Φ. The curve φ (section OA in Fig. 6) determines the values of stresses during a monotonic increase in strain along the section $(0, \varepsilon_o)$. The curve Φ defines stresses during monotonic change in strain along the section $(\varepsilon_o, -\varepsilon_o)$ (section AB in Fig. 6) and along the section $(-\varepsilon_o, \varepsilon_o)$ (section BA in Fig. 6).

The form of the function Φ is similar to the function φ and can be described as follows

$$\Phi(\varepsilon) := 2\varphi\left(\frac{\varepsilon}{2}\right) \tag{15}$$

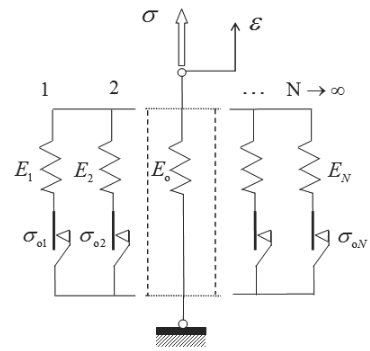

Fig. 5. Rheological scheme of generalized Prandtl model with hardening

Equation (13) can be applied for symmetrical Prandtl model, when in each network limit stresses in tension and compression have the same value $\sigma_{oi}^+ = \sigma_{oi}^- = \sigma_{oi}$. In case of non-symmetrical model exhibiting various plastic limits in tension and compression, the following formula can be applied

$$\Phi(\varepsilon) := (1+\alpha)\,\varphi\left(\frac{\varepsilon}{1+\alpha}\right), \quad \alpha \in \langle 0, 1 \rangle \tag{16}$$

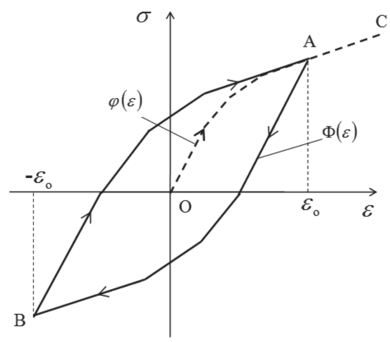

Fig. 6. Hysteretic loop of generalized Prandtl model with hardening

The influence of the value of parameter α on the location of hysteretic loop is shown in Fig. 7.

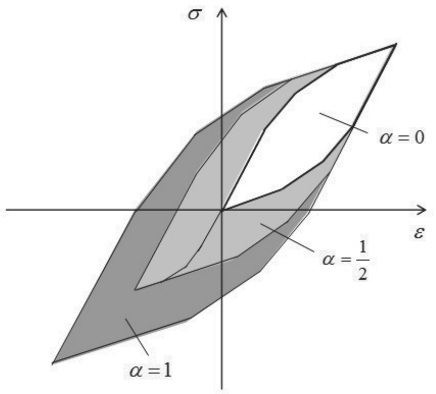

Fig. 7. Hysteretic loops of generalized Prandtl model with hardening for various parameters α

Finally we will formulate the system of constitutive equations for generalized Prandtl model with hardening. In order to do so, we need to use the Hooke's law for elastic parts, the plastic characteristic shown in Fig. 3b and the strain/stress decomposition rule. The procedure needs also the use of so called differential succession to be applied [11]. The algorithm leading to the system of explicit differential equations was explained in [7]. The complete set of relationships is as follows

$$\sigma = E_o\,\varepsilon + \sum_{i=1}^{N} \sigma_i \tag{17a}$$

$$\sigma_i = E_i\left(\varepsilon - \varepsilon_{pl}^{i}\right) \tag{17b}$$

$$\dot{\varepsilon}_{pl}^{i} = \begin{cases} 0 & \text{if} \quad -\sigma_{oi}^{-} < \sigma_i < \sigma_{oi}^{+} \\ 0 & \text{if} \quad \sigma_i = \sigma_{oi}^{+} \text{ and } \dot{\varepsilon} \le 0 \\ 0 & \text{if} \quad \sigma_i = -\sigma_{oi}^{-} \text{ and } \dot{\varepsilon} \ge 0 \\ \dot{\varepsilon} & \text{if} \quad \sigma_i = \sigma_{oi}^{+} \text{ and } \dot{\varepsilon} > 0 \\ \dot{\varepsilon} & \text{if} \quad \sigma_i = -\sigma_{oi}^{-} \text{ and } \dot{\varepsilon} < 0 \end{cases} \tag{17c}$$

6. Identification of composite model

The procedure of identification of material model will be demonstrated using experimental results of cycling tension of fiber glass-reinforced polymer-matrix composite (see Fig. 8).

Figure 9 shows the results of experiment with a marked line corresponding to the functions φ and Φ. The marked line φ was approximated using the following polynomial

$$\varphi(x) := \sum_{i=1}^{5} a_i x^i \tag{18}$$

The parameters a_i of the function defined by Eqn. (18) were obtained using least-squares algorithm implemented in MATLAB software (see Fig. 10). The values obtained are $a_1 = 0.6945$ E4, $a_2 = -1.3895$ E4, $a_3 = 1.8628$ E4, $a_4 = -1.3911$ E4, $a_5 = 0.4156$ E4.

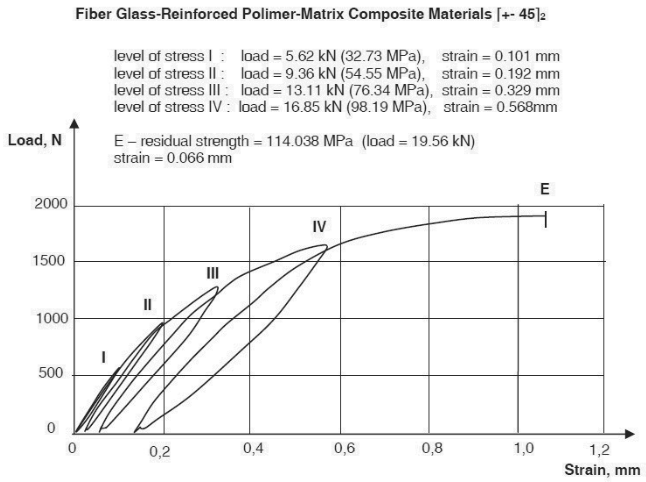

Fig. 8. Result of cyclic loading of the composite

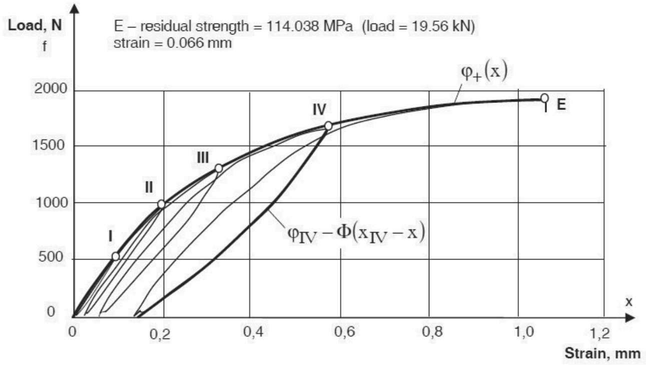

Fig. 9. Result of cyclic loading of the composite with marked curves φ and Φ

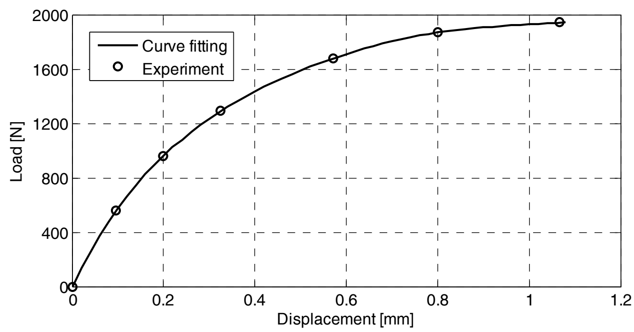

Fig. 10. Result of approximation of the function φ

The same optimization procedure was applied in order to find the value of the parameter $\alpha = 0.2617$ in Eqn. (16). The graphical visualization of the results is shown in Fig. 11.

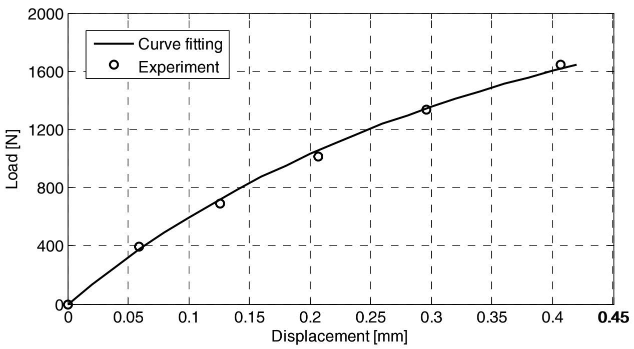

Fig. 11. Result of approximation of the function Φ

In order to simulate the cyclic experiment shown in Fig. 8, the rheological scheme of generalized Prandtl model with finite number of branches will be used. The characteristics of the model are determined by the function φ presented in Fig. 10.

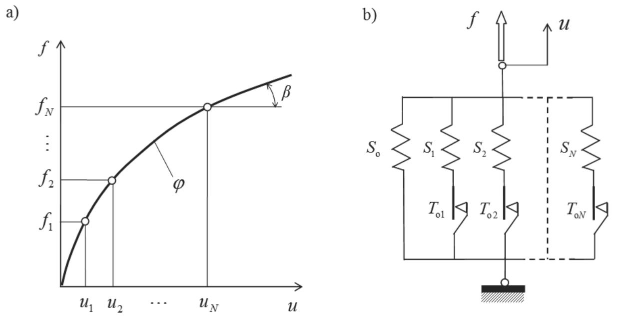

Fig. 12. Identification of rheological 1-D model

The idea of identification is presented in Fig. 12. Based on the values u_i and f_i (see Fig. 12a) we can evaluate the parameters of rheological scheme shown in Fig. 12b. It should be emphasized that the hysteretic loop we will simulate (see Fig. 8) is defined on the plane displacement – force $u - f$. Thus, the parameters shown in Fig. 12b are spring stiffnesses S_i and slider limit forces T_{oi}.

The identification procedure leads to the following equations relating the points (u_i, f_i) taken from approximation of experiment and the parameters of rheological scheme S_i and T_{oi}.

$$
\begin{aligned}
&S_o = \tan\beta \\
&S_N = \frac{f_N - f_{N-1}}{u_N - u_{N-1}} - S_o \\
&S_i = \frac{f_i - f_{i-1}}{u_i - u_{i-1}} - S_o - \sum_{j=i+1}^{N} S_j \quad \text{for } i = N-1, N-2, \dots 1 \\
&T_{oi} = S_i u_i
\end{aligned}
\tag{19}
$$

We will simulate the experiment shown in Fig. 8, solving the following system of differential equations (compare with Eqns. 15)

$$
f = S_o u + \sum_{i=1}^{N} f_i \tag{20a}
$$

$$
f_i = S_i \left(u - u_{pl}^i\right) \tag{20b}
$$

$$
\dot{u}_{pl}^i = \begin{cases} 0 & \text{if} \quad -T_{oi}^- < f_i < T_{oi}^+ \\ 0 & \text{if} \quad f_i = T_{oi}^+ \text{ and } \dot{u} \le 0 \\ 0 & \text{if} \quad f_i = -T_{oi}^- \text{ and } \dot{u} \ge 0 \\ \dot{u} & \text{if} \quad f_i = T_{oi}^+ \text{ and } \dot{u} > 0 \\ \dot{u} & \text{if} \quad f_i = -T_{oi}^- \text{ and } \dot{u} < 0 \end{cases} \tag{20c}
$$

The number of rheological scheme's networks assumed was $N = 10$. The fixed-step Runge-Kutta algorithm was used to solve differential equations. The results of simulations are presented in Fig. 13. Comparing the Figs. 8 and 13 shows good agreement between experiment and simulation.

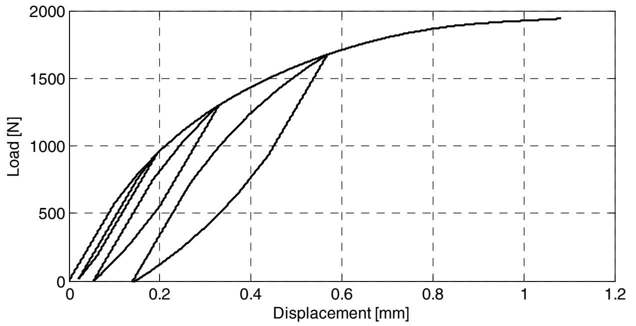

Fig. 13. Results of numerical simulations for 10-network generalized Prandtl model

7. Conclusions

The original value of the paper is the proposed procedure of identification of material model exhibiting plastic anisotropy based on generalized Prandtl rheological scheme. As the result, the system of constitutive relationships has explicit differential form, easy for numerical implementations.

It was proved in the paper that the obtained constitutive relationships can describe the phenomenon of plastic anisotropy. The illustrating example was demonstrated for fiber glass-reinforced polymer-matrix composite. The comparison between experimental results and computer simulations shows the validity of the model.

Using the method of constitutive modelling of elastoplastic properties of materials presented in the paper, it is possible to carry out computer simulations solving non-linear differential equations for both static and dynamic loadings.

The computer simulations concentrate on one-dimensional problem. It is suggested for future investigations to implement three-dimensional constitutive model. Such an implementation may be conducted within FEM codes ABAQUS or ANSYS [14,15].

Acknowledgements

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