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Elastic metamaterials: some initial experiences with the Milton-Willis structure based on a rapid prototyped model and a numerical analysis

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Analysis and modelling

ABSTRACT

Elastic metamaterials attract quite a bit of research attention of late. Theoretical models such as those proposed by Milton and Willis suggest possibility of physically realizing the structures leading to elastic metamaterials. The main stumbling block being the limitations of fabrication methods, considering the complex geometries and material requirements. Rapid prototyping and of-late rapid manufacturing offer solutions for the production of physical shapes of unlimited geometrical complexities direct from digital files, employing a variety of materials. Considering the requirements of metamaterial structures and the capabilities of rapid manufacturing, the need to bring these two together has been envisioned, and the experimental and numerical work presented in this paper is an initial step towards this goal.

Keywords: Elastic metamaterials; Milton-Willis material; Rapid prototyping

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1. Introduction

Invisibility cloaking has been the research focus in metamaterials field for the past few years. This research field started through the work of Smith et al, where they proposed a method to bend electromagnetic waves using gradient refractive index [1]. Based on this study, Leondhardt and Pendry showed that electromagnetic invisibility cloaking can be realized in micro wave range [2,3].

Current research in metamaterials is also focused on exploring invisibility cloaking in various fields such as the elastic fields. Quite a few researches are looking at the possibility of guiding elastic waves such as those generated by seismic activities. In one of the studies, Milton, Briane, and Willis found that principle of transformation based cloaking can be applied to the continuum elasto-dynamic concept [4].

However in order to do so, a new material with unusual properties need to be constructed (referred as elastic metamaterials). By comparing maxwell's equation and continuum elastodynamics, they found that a relation between the two concepts can be made as long as the new material formed has its density as a function of the frequency [5].

Moreover, stress strain relation in the material has to behave different compared to the natural material. The stress in the elastic metamaterials has to depend not only on strain but also on velocity. Also, momentum density in this material has to be coupled not only with velocity but also with displacement gradient through strain. Cummer and Schurig investigated the necessity of having anisotropic density in the elastodynamics metamaterials [6].

Subsequently, Milton and Willis came up with a design that satisfies the aforementioned properties [7]. As shown in Fig. 1, the Milton-Willis material consists of a network of springs connected to masses of different densities (dark and white circles).

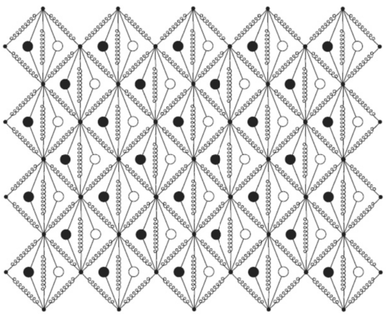

Fig. 1. Milton-Willis material [7]

Fig. 2 shows the unit cell configuration of the Milton-Willis material. Points E and F are solid masses. The length and height of the unit cell is 2 h. Link ED, DF, FB, and EB are rigid links [7]. Each joint of this rigid link must allow planar movement along x1 and x2 axes. Springs are in their natural condition. The stiffness of the spring must be considerably less compared to that of the rigid links.

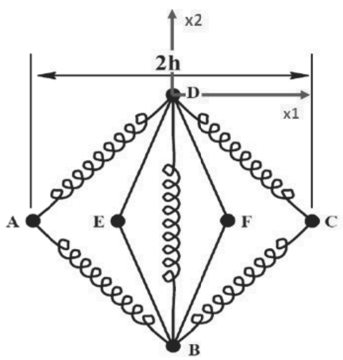

Fig. 2. Unit cell of Milton-Willis material

Mass E is different to mass F; one of them has to be significantly larger than the other. Also they are not lump masses. Each of them in turn has a configuration as depicted in Fig. 3 below, where a solid mass is resting in the middle of a rigid ring structure and connected through four equidistant springs [7].

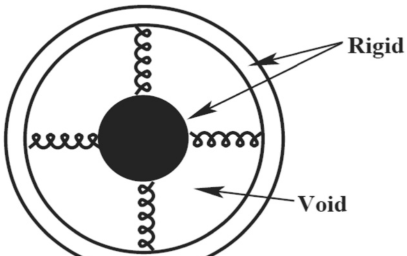

Fig. 3. Solid mass configuration [7]

Ideally, this solid mass is small in size such that it can be treated as a point mass. In fact, all the mathematical derivation is done assuming the whole structure to be available in the smallest scale possible. For example, in the analytical derivation, the limit of h is often assumed to approach 0. However, to have this arrangement at this stage is not feasible and economical. Followings are the final expressions for momentum density and stress of the model as derived by Milton [7].

In the limit h→0, the stress and momentum matrix are formed as the following:

$$\begin{pmatrix}\boldsymbol{\sigma}\\ \boldsymbol{p}\end{pmatrix}=\begin{pmatrix}\mathrm{C} & \boldsymbol{S}\\ \boldsymbol{D} & \boldsymbol{\rho}\end{pmatrix}\begin{pmatrix}\nabla\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix}$$

$$\boldsymbol{p}=\begin{pmatrix}p1\\ p2\end{pmatrix},\boldsymbol{v}=\begin{pmatrix}v1\\ v2\end{pmatrix},$$

$$\nabla\boldsymbol{u}=\begin{pmatrix}\frac{\partial u1}{\partial x1}\\ \frac{\partial u2}{\partial x1}\\ \frac{\partial u1}{\partial x2}\\ \frac{\partial u2}{\partial x2}\end{pmatrix},\boldsymbol{\sigma}=\begin{pmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{12}\\ \sigma_{22}\end{pmatrix} \tag{1}$$

where:

- is the stress vector which consists of principal stresses acting on infinitesimally small particle in the model such as σ_{11}, and so on.
- is the momentum density vector (p_1 momentum density in direction x1 and p_2 momentum density in direction x2). is density matrix of the model, is elasticity tensor of the model, is the velocity vector and (v1 velocity in direction x1 and v2 velocity in direction x2, is frequency of the vibration). While, is displacement gradient.
- and **D** are the third order sensors and their expressions are:

$$\mathbf{S}=\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & -i\omega mc\\ -\frac{i\omega m}{c} & 0\end{pmatrix}$$

$$\boldsymbol{D}=\begin{pmatrix}0 & 0 & 0 & -i\omega m/c\\ 0 & 0 & -i\omega mc & 0\end{pmatrix} \tag{2}$$

As can be seen from the series of mathematical expressions above, stress is coupled directly with velocity and the momentum density is directly related with the displacement gradient, satisfying the elastic metamaterials envisioned by Milton [7].

2. Fabrication of Milton-Willis structure

The material presented by Milton & Willis is a theoretical material, in which manufacturability of the material is not considered. An attempt is made in this paper to design and fabricate the Milton-Willis material as depicted in Fig. 4 below.

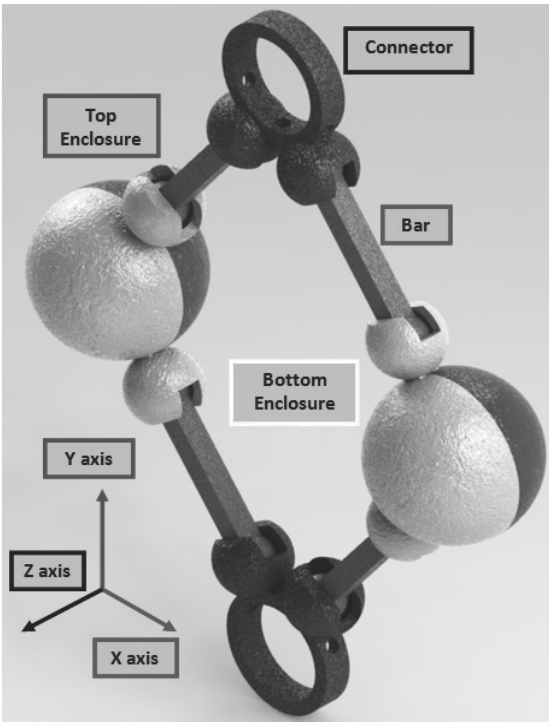

Fig. 4. Designed Milton-Willis material configuration

All parts of the material shown above are rigid, and the material consists of 4 different parts which are named as shown in Fig. 4. The rigid parts are made of nylon; solid masses used are lead and steel balls. Instead of using springs, silicone gel is used to support the rigid structure and the solid masses. As mentioned briefly above, rigid parts of the material need to be considerably stiffer than the elastic parts. This is verified by performing tensile tests on both materials (nylon and silicone). Following table lists the material used and relevant properties.

Eight ball joints are used in this material structure. However rotation about X and Y axis is limited, meanwhile rotation about Z axis is allowed. Slots at the joint enclosure allow the Z axis rotation, while small height clearance between bar and the joint enclosure limits the X and Y axis rotation. This range of motion follows the requirement of the model as described in the literature where model motion is mainly in the XY plane.

Table 1.
Material properties

Material	Properties	Value
Steel	Elastic Modulus	2×10^9 Pa
	Poisson's ratio	0.3
	Density	7850 kg/m^3
Silicone	Elastic Modulus	1.66×10^5 Pa
	Density	2330 kg/m^3

The rigid structure is made by using the selective laser sintering (SLS) method. The choice of SLS over other plastic manufacturing methods is due to the ease of manufacturing the complex shape direct from CAD files, without any intermediate tooling. Generally, the properties of parts produced by SLS are relatively inferior compared to other traditional methods. However, considering the complexity, the smallness of the scale and the ability to produce assemblies as one unit would limit suitability of traditional methods. Considering the rapid changes taking place in rapid prototyping and manufacturing, there is an ever increasing list of candidate materials and commercially successful manufacturing techniques, that will eventually make metamaterials research to be effectively married to rapid manufacturing technologies.

As mentioned briefly above, silicone gel is used to replace the springs in the structure. This replacement is reasonable as long as the silicone gel can provide elastic support for the rigid structure and the solid masses. Silicone gel is placed inside the half sphere cavity at both locations in the unit cell. After the silicone gel is well distributed inside the cavity, the solid mass is carefully placed at the centre of bottom enclosure. While half the cylinder is resting on a silicone gel, the other half is covered from the top by the cap. Fig. 5 below depicts the assembly process used to put the solid-mass nodes together.

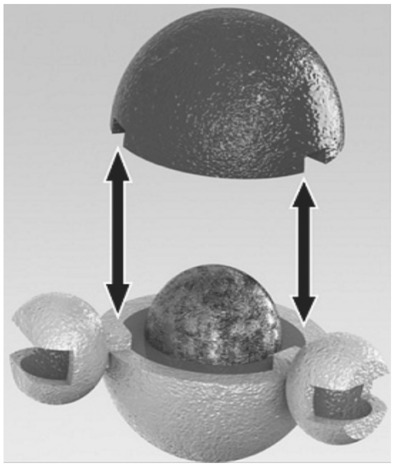

Fig. 5. Silicone support for solid mass

Subsequently, the assembled structure is placed inside an aluminum-wood mould. Then the silicone gel is distributed evenly until the rigid structure is fully immersed inside the silicone gel. The immersed structure is left to bond and solidify with the silicone gel. The final product is a slab of hardened silicone gel with the assembled structure embedded inside as shown in Fig. 6.

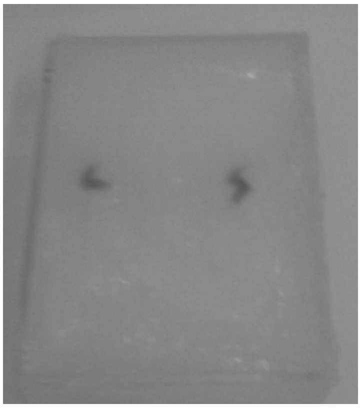

Fig. 6. Finished product of Milton-Willis material

3. FEA of the Milton-Willis material designed

The analysis starts with modal analyses on both solid mass (lead ball and steel ball) and the overall structure. The main purpose of the modal analysis is to obtain the natural frequency of the system. Ideally, natural frequency of the overall structure matches with the natural frequency of both solid masses. The matching natural frequency is then used as operating frequency for the dynamic analysis of the structure. The objective of the dynamic analysis is to obtain the stress-strain relation and the dynamic relation of the structure (momentum-velocity relation).

Modal analysis on the solid mass is done on the solid mass itself and the elastic support (silicone gel) surrounding it Boundary condition is set at the outer surface of the silicone layer in which no displacement is allowed at the outer surface. Inner surface of the silicone layer which makes a contact with the solid mass is also regulated. The silicone gel is allowed to have frictional contact with the solid mass. Thus deformation and volumetric displacement between silicone and solid mass are possible to occur. Solid mass is set as a rigid (non deformable) member while silicone gel is set as a flexible member. Harmonic vibration varying from 1-2 GHz is induced on the solid mass and its support. Mode shapes and corresponding modal frequency are then obtained.

Among all results, one result from each solid mass shows resemblance with each other. The modal frequency of the steel ball is found to be 25.73 Hz and the modal frequency of the lead ball is found to be 24.85 Hz. Three percent difference between the two frequencies can be considered as insignificant. Their mid value of 25.29 Hz is taken as the representative value for later comparison.

As mentioned before, modal analysis is also performed on the overall structure to obtain its natural frequency. The natural frequency is then compared with the natural frequency of the solid mass. Based on this comparison the operating frequency for the dynamic analysis is decided. Modal analysis for the overall structure involves the nylon structure with the solid mass inside them and also silicone gel that covers the nylon structure and the solid mass.

Frictional contact is defined between the nylon structure and the silicone gel that covers them. The nylon is defined as a rigid member, meanwhile silicone gel is allowed to deform freely. Unlike in the modal analysis for solid mass, the outer surface of the silicone gel in this analysis is allowed to deform freely. Also, silicone is treated as anisotropic material to realistically emulate the real life condition.

A harmonic vibration ranging from 0-2 GHz is applied to the overall structure. Mode shapes and corresponding frequency are obtained. Among all results obtained, one result possesses modal frequency relatively close to the natural frequency of the solid mass. Fig. 7 shows the aforementioned result.

The mode shape of the structure has a natural frequency of 24.781 Hz which is relatively close to the natural frequency of the solid mass of 25.29 Hz. The difference of 3.6% between them can be considered insignificant. This mode shape shows that the two solid masses move in accordance in Y axis. Their movement is mainly caused by the large displacement of the nylon structure which is displaced vertically in Y axis. The steel ball is displaced more compared to the lead ball. This is as expected since lead is more dense than steel. The frequency of 24.781 Hz is then used as the operating frequency for the dynamic analysis. Natural frequency is used as the operating frequency to amplify the displacement effect.

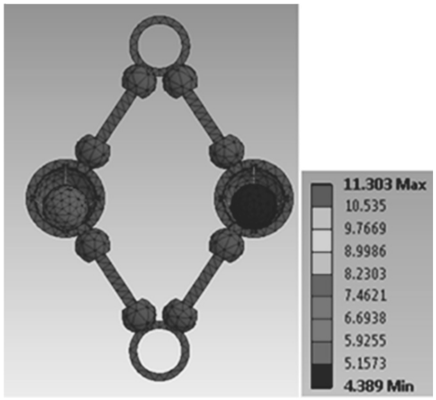

Fig. 7. Modal analysis result of the overall Milton-Willis structure

As mentioned above, the objective of the dynamic analysis is to understand the dynamic behaviour of the structure (momentum-velocity relation) and the stress-strain relation of the structure. The structure is loaded with a sinusoidal wave with 0.2 mm peak to peak Y axis displacement at 24.781 Hz at the top of the structure. The bottom part of the structure is fixed. 4 Nodal displacements and forces are monitored at the two solid masses and the top and bottom part of the structure.

Control volume is set around the outer surface of the silicone gel. Fig. 8 below shows the control volume set up.

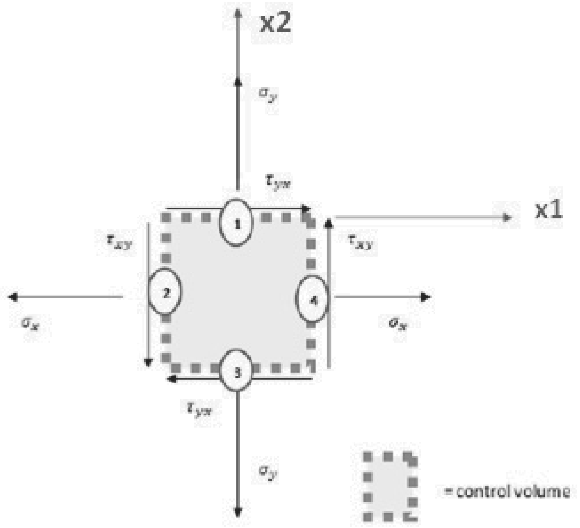

Fig. 8. Control volume analysis

Imagine that the structure rests inside the green boundary. Only planar case is considered in the simulation (x1 x2 plane). Axial forces are found along the x1 and x2 axes of the control volume. Shear forces are also found at the periphery of the control volume. Stress-strain and momentum-velocity relation can be obtained based on the parameters monitored and geometrical data of the structure. Both relations can be arranged into matrix form as shown below, with stress and momentum at one side and related by a constant matrix with momentum and velocity at the other side.

$$[A] = [X] * [B]$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{bmatrix} \begin{bmatrix} u_{x1\,x1} \\ u_{x1\,x2} \\ u_{x2\,x1} \\ u_{x2\,x2} \\ V_{x1} \\ V_{x2} \end{bmatrix} \tag{3}$$

After stress, strain, momentum and velocity are fully defined. The constant can be calculated. This constant is the indicator on how the structure behaves dynamically. Based on the simulation result the constant matrix yields the following result.

$$[X] = \begin{bmatrix} 2.9301 & 0 & 0 & 0 & 0 & 0 \\ 2.65 & 0 & 1.21 & 0 & 0 & 0 \\ -2.76 & 0 & 0 & 0 & 0 & 0 \\ -2.3676 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.9743 & 0 \\ 0 & -0 & 0 & 0 & -0.1331 & -4.2108 \end{bmatrix} \tag{4}$$

In a normal material, σ_{11} and σ_{22} depend on the axial strain that is indicated with the displacement gradient in this case (u_{x1x1}, u_{x2x2}). However in this structure σ_{11} is only dependent on u_{x1x1}. While σ_{22} is behaving abnormally as it only depends on shear strain. σ_{12} is dependent on shear strain which is indicated by u_{x1x2}, u_{x2x1} in this case. However its counterpart σ_{21} is behaving out of ordinary where no dependency with shear strain is evident. P_1 is the momentum density in x1 direction, meanwhile p_2 is momentum density in x2 direction which normally depends on velocity. This is clearly evident in the silicone structure where p_1 and p_2 are highly dependent on velocity.

Based on the constant matrix X shown above, it can be concluded that the silicone model is behaving as a normal material. The proposed coupling of stress and velocity and momentum and strain are not evident in this structure. However an encouraging result from this simulation is the mechanism in the structure is behaving normally. The proposed relation is not evident due to the size the structure has. Ideally, as the size is reduced the structure would show the proposed stress and momentum behaviour.

4. Conclusions

Elastic metamaterials structure proposed by Milton-Willis is being realized and constructed as part of the work presented in this paper. The derivations proposed in Milton's paper were made based on the assumptions of ideal conditions around the material's size being infinitesimal and different parts of structures behaving as expected. However, the real model will not behave exactly like this, considering that it has a finite size in reality.

FEA was done on the feature of the built Milton-Willis model. First modal analysis was performed to obtain the natural frequency of solid mass and the overall structure. It was found that the natural frequency of the solid mass matched with natural frequency of the overall structure. This is as expected since mass of the overall structure is basically constituted of the solid masses. The natural frequency is then used as the operating frequency for the dynamic analysis where the overall structure is displaced by a sinusoidal waveform operating at the natural frequency of the overall structure. Natural frequency is chosen as the operating frequency to allow the dynamic effect to be seen clearly. Based on the dynamic analysis, the material shows normal material behaviour where stress is only dependent on strain and momentum is only dependent on velocity.

The built Milton-Willis model moved well dynamically. All joints in rigid part are moving well without too much resistance. Even though the material doesn't show the unusual dynamic behaviour as expected, the smooth movement of the rigid structure inside the silicone slab is encouraging. Our prediction is the results will get more encouraging, if the size of the overall structure is reduced. Also, stiffness difference between elastic support and rigid parts must be increased. An experimental evaluation of this structure is currently being undertaken.

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