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An analysis of discrete-continuous mechanical systems with conjugations

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Analysis and modelling

ABSTRACT

Purpose: The main purpose of this work is developing a methodology, using non-classical methods, of modelling the complex mechanical systems with the continuous and discrete-continuous distribution of parameters. A simple task of dynamics can be solved by using this method, without limitations deriving from the type and number of the elements of a mechanical system.
Design/methodology/approach: By using the non-classical methods of modelling, it was possible to develop a method of determining the matrices (flexibilities) of multi-link vibration mechanical systems with the continuous distribution of parameters that are able to perform longitudinal and flexural vibrations. The method is focused on broadening graphs method by mechanical systems and improving their description and design methods so that the mathematical formalism can reflect the essence of the problem involved in the designation of dynamic characteristics of such systems.
Findings: The knowledge of the dynamic characteristics of a system determined for any inputs and outputs in form of kinematic and dynamic excitations is underlying the determination of frequency characteristics of the class of the systems under consideration.
Research limitations/implications: The class of the systems considered refers to investigating into the dynamic and vibration characteristics of mechanical systems with the discrete-continuous distribution of parameters performing small vibrations around the adopted state of equilibrium.
Practical implications: The presented method of this study is that the main point can be the introduction to e.g. additional kinematic excitations in form of a function of speed and accelerations or extending the method presented to cover the investigation of non-linear systems.
Originality/value: The modelling and analysis of discrete–continuous vibration systems with conjugations using the non-classical method is a more general approach as compared to modelling and analysis in classical terms.
Keywords: Graphs; Vibrating mechanical systems

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1. Introduction

An important field playing a major role in mechanical engineering is the field dealing with the vibrations of systems. The issue of the vibrations of physical systems is very important for practical implications. For this reason it is necessary to conduct research the main aim of which is to understand the phenomena related to the vibrations of complex systems and also to determine the influence of a change to the parameters of the system on its behaviour during vibrations. A phenomenon of vibration occurs in each mechanical object, therefore, it is the key issue both, in the area of purely practical concepts, scientific and research concepts as well as in the area of didactics. The primary purpose of a dynamic analysis is to understand, predict and, sometimes, correct the behaviour of an object subjected to dynamic and kinematic interactions. Vibration systems can be generally grouped into: systems with the continuous distribution of parameters, systems with the discrete distribution of parameters and systems with the discrete and continuous distribution of parameters. This grouping is of mathematical character and takes the research methodology into account. This is why information acquired from the simplified models of real systems is used in many cases only. Nearly every machine nowadays, in the era of advanced technologies, incorporates the components being mechanical or electrical vibration systems. For this reason an analysis of mechanical systems is still an important stage of designing and operating the technical systems. Considering that mechanical objects are more and more complicated in constructional and functional terms, difficulties often arise when applying a classical mathematical apparatus for creating and solving the relevant systems of the differential equations of motion. The analytical solving of differential equations, referred to in the literature as "*classical methods*", representing a mathematical model, is time-consuming or may be impossible, at all. When creating a physical model and then a mathematical model of the analysed mechanical object, versatile knowledge on its properties and the processes and relationships occurring in the model is indispensable. The knowledge must be supported by the knowledge of the laws of physics (mechanics, electrical engineering or thermodynamics) determining the behaviour of the system. When analysing a system made up of multiple subsystems, a great deal of work-consuming and time-consuming activities is required to determine dynamic characteristics. Where it is necessary to modify the structure of the system, the differential equations of motion have to be formulated each time from the beginning or the matrices of stiffness, dampening and others have to be constructed anew. Non-classical methods are helpful then. The methods are often characterised by a high degree of algorithmisation which makes it easier to implement them in computer calculation systems, especially when determining dynamic characteristics. They also enable to present the structure of the modelled system graphically [1-9, 16-19].

This article concentrates on developing a methodology – using non-classical methods – of modelling the complex mechanical systems with the continuous and discrete-continuous distribution of parameters. The class of the systems considered refers to investigating into the dynamic and vibration characteristics of mechanical systems with the discrete-continuous distribution of parameters performing small vibrations around the adopted state of equilibrium.

The knowledge of the dynamic characteristics of a system determined for any inputs and outputs in form of kinematic and dynamic excitations is underlying the determination of frequency characteristics of the class of the systems under consideration.

2. Subsystems with the continuous distribution of parameters

The method, presented in this chapter, of determining a stiffness matrix of multi-link systems with the continuous distribution of parameters performing longitudinal or torsional, represents an original solution serving to determine the dynamic characteristics of the considered class of systems. The process of determining a stiffness matrix has been shortened substantially by applying this method for analysing mechanical systems with the continuous distribution of parameters where a system constructed of multiple links is analysed.

2.1. Systems with the continuous distribution of parameters vibrating longitudinally

The subject of the considerations are multi-section, mechanical rod systems vibrating longitudinally in form of models with their parameters distributed in a continuous manner and with sectionally constant cross-section. Two basic pools of physical values are used for describing the adopted model. ${}_1S$ and ${}_2S$ (where the ${}_1S$ – is a pool of generalised system displacement values; the ${}_2S$ – is a pool of the generalised values of forces). The relationships between the amplitudes of generalised forces ${}_2s_j \in {}_2S$ and generalised displacements ${}_1s_i \in {}_1S$ are expressed by applying a concept of dynamic flexibility Y_{ij}, i.e. an amplitude, bearing a relevant sign, of generalised displacement in the direction of the *i*-th generalised coordinate, caused by a generalised force in form of a harmonic function with the unit amplitude, corresponding to the *j*-th of this generalised coordinate, i.e.:

$$ {}_1s_i = Y_{ij}\ {}_2s_j \tag{1}$$

where: ${}_2s_j = e^{a\omega t}$, $a = \sqrt{-1}$, ω – frequency.

When considering the longitudinal vibrations of the free rod (Fig. 1), the amplitudes of displacements ${}_1s_1^{(i)}$ and ${}_1s_2^{(i)}$ assigned to its extreme positions are used for description.

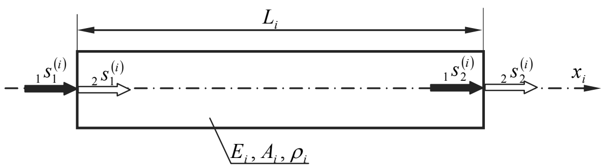

Fig. 1. A continuous and limited model of a longitudinally vibrating rod

The meanings of the symbol in Fig. 1 are:

${}_1s_1^{(i)}$, ${}_1s_2^{(i)}$ – the values of linear displacements of the rod ends,

${}_2s_1^{(i)}$, ${}_2s_2^{(i)}$ – the values of linear forces ,

E_i – elastic modulus,

A_i – the rod cross-section field,

ρ_i – the rod material density.

The pools of the generalised values of displacements and the values of forces of the considered link assume the following form:

$$\begin{cases} {}_1S = \{{}_1s_1, {}_1s_2\}, \\ {}_2S = \{{}_2s_1, {}_2s_2\}. \end{cases} \tag{2}$$

The relationships between the values of generalised displacements and the values of generalised forces for a single link in the system are recorded as follows:

$$\begin{bmatrix} {}_1s_1 \\ {}_1s_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} {}_2s_1 \\ {}_2s_2 \end{bmatrix} \tag{3}$$

By assuming that the Y flexibility matrix is non-singular, they have been converted to the following form:

$$\begin{bmatrix} {}_2s_1 \\ {}_2s_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} {}_1s_1 \\ {}_1s_2 \end{bmatrix} \tag{4}$$

or

$${}_2S = Z\ {}_1S \tag{5}$$

Digital indices were replaced by literal indices to simplify the notation:

$$\begin{bmatrix} {}_2s_1 \\ {}_2s_2 \end{bmatrix} = \begin{bmatrix} z_a & z_c \\ z_d & z_b \end{bmatrix} \cdot \begin{bmatrix} {}_1s_1 \\ {}_1s_2 \end{bmatrix} \tag{6}$$

The subject of the considerations is the K structure of a mechanical system, with the continuous distribution of parameters and with sectionally constant cross-section and vibrating longitudinally (Fig. 2), defined as:

$$K = \langle S, R_z \rangle \tag{7}$$

where:
S is a pool of generalised variables (the values of displacement and values of forces), R_z – a bilateral relation referred to as follows:

$$\langle {}_1s_i, {}_2s_j \rangle \in S \underset{\gamma_P}{\vee} \{\gamma_p({}_1s_i) = {}_2s_j\} \wedge {}_1s_i, {}_2s_j \in S \tag{8}$$

where:
$\gamma_p \in \Gamma$ $(i, j = 1, 2, \ldots, n;\ p = 1, 2, \ldots, \text{card } \Gamma)$, Γ – a pool of γ_p functions determined with the S pool.

The form of a matrix flow graph representing the structure of the considered system (Fig. 2) was obtained by making the mappings and assignments described in [11, 12]

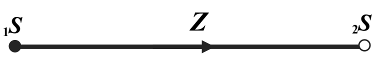

Fig. 2. Matrix signal flow graph

where:
${}_2S$ – a column matrix of the generalised forces of the system,
${}_1S$ – a column matrix of the generalised displacement of the system,
Z – a stiffness matrix of the considered system.

It is not necessary to draw a flow graph in order to determine the stiffness matrix of the systems made up of the l – rod elements. It is enough to use the method of summing stiffness matrices for the next links [11, 12], recorded with the relationship (9). This method makes it easier to determine the stiffness matrix of the considered system, therefore, it allows for the determination of the system's flexibility anywhere in the abrupt change of section.

$$Z = \begin{bmatrix} Z_a^{(1)} & Z_c^{(1)} & 0 & 0 & \cdots & 0 \\ Z_d^{(1)} & Z_b^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & Z_a^{(2)} & Z_c^{(2)} & 0 & \cdots & 0 \\ 0 & Z_d^{(2)} & Z_b^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \ldots$$
$$+ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_a^{(l)} & Z_c^{(l)} \\ 0 & 0 & 0 & 0 & Z_d^{(l)} & Z_b^{(l)} \end{bmatrix} \tag{9}$$

The final result of the (9) relationship is the $\boldsymbol{Z}$ stiffness matrix dimensioned $[ns\text{-}1] \times [ns\text{-}1]$, where: n – the number of subsystems of the continuous system, s – the number of coordinates of the generalised system.

Ultimately, the stiffness matrix of a multi-link mechanical system vibrating longitudinally assumes the following form:

$$\mathbf{Z} = \begin{bmatrix} Z_a^{(1)} & Z_c^{(1)} & & & & & & 0 \\ Z_d^{(1)} & Z_b^{(1)} + Z_a^{(2)} & Z_c^{(2)} & & & & & \\ & Z_d^{(2)} & Z_b^{(2)} + Z_a^{(3)} & Z_c^{(3)} & & & & \\ & & Z_d^{(3)} & Z_b^{(3)} + \cdots & & & & \\ & & & & \ddots & & & \\ & & & & Z_d^{(l-2)} & Z_b^{(l-2)} + Z_a^{(l-1)} & Z_c^{(l-1)} & \\ & & & & & Z_d^{(l-1)} & Z_b^{(l-1)} + Z_a^{(l)} & Z_c^{(l)} \\ 0 & & & & & & Z_d^{(l)} & Z_b^{(l)} \end{bmatrix} \tag{10}$$

The stiffness matrix of the l – link systems (10) obtained constitutes a basis for analysing the system further, i.e. for determination of the $\boldsymbol{Y}$ flexibility matrix of the rod system vibrating longitudinally. By assuming that the $\boldsymbol{Z}$ matrix is non-singular, the $\boldsymbol{Y}$ flexibility matrix is the inverse of the $\boldsymbol{Z}$ stiffness matrix.

$$Y_{ij} = (-1)^{i+j} \frac{\det Z_{ij}}{\det Z} \tag{11}$$

where:
Y_{ij} – an element entered in the i–th row and in the j–th column of the system flexibility matrix,
$\det Z$ – a determinant of the stiffness matrix,
$\det Z_{ij}$ – a determinant of the stiffness matrix without the i–th row and the j–th column.

The appropriate flexibilities of the longitudinally vibrating rods have to be provided to obtain the sought dynamic flexibilities with the specific boundary conditions. Various boundary conditions imposed on the ends of links need to be taken into account to determine the dynamic characteristics of real mechanical systems. The dynamic stiffness matrices of the system do not have to be modified when changing the boundary conditions. It is sufficient to provide the relevant flexibilities of the system for the stiffness matrix of the system.

2.2. Systems with the continuous distribution of parameters vibrating flexurally

The subject of the considerations are flexurally vibrating, multi-section rod systems with their parameters distributed continuously and with sectionally constant cross-section. It is assumed that the vibrations of the link occur in a single plane. A movement of any link element is determined by two coordinates. The coordinates include: bending ${}_1s_i = y(x, t)$ and bending angle ${}_1s_i = \dfrac{\partial y(x,t)}{\partial x}$ (in any element of the link). It is also assumed that the link may be exerted with a generalised force (a harmonically-variable force or a harmonically-variable bending moment). This means that the notion of dynamic flexibility is considered in this case in a broader sense as if it were done for longitudinal vibrations. Dynamic flexibility is recorded as a relationship between bending and a bending moment or a lateral force or between a bending angle and a bending moment or a lateral force. This relationship, which is in line with the definition of dynamic flexibility, can be recorded with words, using the following diagram:

$$\begin{pmatrix} \text{amplitude of bending} \\ \text{amplitude of section rotation angle} \end{pmatrix} = \begin{pmatrix} \text{dynamic} \\ \text{flexibility} \end{pmatrix} \cdot \begin{pmatrix} \text{amplitude of lateral force} \\ \text{amplitude of bending moment} \end{pmatrix} \quad (12)$$

Displacement amplitudes (${}_1s_1$ and ${}_1s_3$) and section rotation angles (${}_1s_2$ and ${}_1s_4$) assigned to the extreme points are used for considering the flexural free vibrations of the link presented in Fig. 3. It is considered that the counter clockwise section torsional angle is positive.

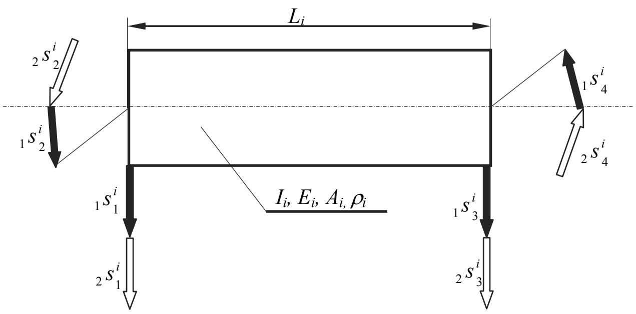

Fig. 3. Continuous and finite model of a flexurally vibrating link

The meanings of the symbol in Fig. 3 are:

I_i – moment of inertia of cross-section,

E_i – elastic modulus,

A_i – the rod cross-section field,

ρ_i – link material density,

${}_1s^i_1$, ${}_1s^i_3$ – linear displacements of the link ends,

${}_1s^i_2$, ${}_1s^i_4$ – angular displacements of the link ends,

${}_2s^i_1$, ${}_2s^i_3$ – values of shearing forces moments,

${}_2s^i_2$, ${}_2s^i_4$ – values of shearing bending moments.

The relationships between the generalised displacements of the rod ends and the forces causing such displacements are recorded in the following form by inversing the $\boldsymbol{Z}$ matrix of link stiffness.

$$\begin{cases} {}_1S = \{{}_1s_1, {}_1s_2, {}_1s_3, {}_1s_4\}, \\ {}_2S = \{{}_2s_1, {}_2s_2\ {}_2s_3, {}_2s_4\}. \end{cases} \quad (13)$$

The subject of the considerations is the K structure of a mechanical system, with the continuous distribution of parameters and with sectionally constant cross-section and vibrating longitudinally (Fig. 3), defined as:

$$K = \langle S, R_z \rangle \quad (14)$$

where:

S is a pool of generalised variables (the values of displacement and values of forces), R_z – a bilateral relation referred to as follows:

$$\langle {}_1s_i, {}_2s_j \rangle \in S \vee_{\gamma_P} \{\gamma_p({}_1s_i) = {}_2s_j\} \wedge {}_1s_i, {}_2s_j \in S \quad (15)$$

where:

$\gamma_p \in \Gamma$ $(i, j = 1, 2, \ldots, n;\ p = 1, 2, \ldots, \text{card}\ \Gamma)$, Γ – a pool of γ_p functions determined with the S pool.

The form of a matrix flow graph representing the structure of the considered system (Fig. 4) was obtained by making the mappings and assignments described in [*]

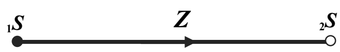

Fig. 4. Matrix signal flow graph

where:

${}_2S$ – a column matrix of the generalised forces of the system,

${}_1S$ – a column matrix of the generalised displacement of the system,

Z – a stiffness matrix of the considered system.

The relationships between the values of generalised displacements and the values of generalised forces for a single link in the system are recorded as follows:

$$\begin{bmatrix} {}_2s_1 \\ {}_2s_2 \\ {}_2s_3 \\ {}_2s_4 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} \cdot \begin{bmatrix} {}_1s_1 \\ {}_1s_2 \\ {}_1s_3 \\ {}_1s_4 \end{bmatrix} \quad (16)$$

or

$$ {}_2S = Z\ {}_1S \tag{17} $$

The relationships between the generalised displacements of the rod ends and the forces causing such displacements are recorded in the following form by inversing the ***Z*** matrix of link stiffness.

$$ \begin{bmatrix} {}_1s_1 \\ {}_1s_2 \\ {}_1s_3 \\ {}_1s_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \cdot \begin{bmatrix} {}_2s_1 \\ {}_2s_2 \\ {}_2s_3 \\ {}_2s_4 \end{bmatrix} \tag{18} $$

or

$$ {}_1S = Y\ {}_2S \tag{19} $$

Digital indices were replaced by literal indices to simplify the notation:

$$ \begin{bmatrix} {}_1s_1 \\ {}_1s_2 \\ {}_1s_3 \\ {}_1s_4 \end{bmatrix} = \begin{bmatrix} Y_a & Y_e & Y_s & Y_t \\ Y_f & Y_b & Y_k & Y_l \\ Y_m & Y_n & Y_c & Y_g \\ Y_o & Y_p & Y_h & Y_d \end{bmatrix} \cdot \begin{bmatrix} {}_2s_1 \\ {}_2s_2 \\ {}_2s_3 \\ {}_2s_4 \end{bmatrix} \tag{20} $$

As generalised displacements are not homogenous (linear displacements or bending angles), flexibilities are expressed with different units:

- m/N – in case of flexibility Y_a, Y_s, Y_m, Y_c,
- m/Nm – in case of flexibility Y_e, Y_t, Y_n, Y_g,
- rad/N – in case of flexibility Y_f, Y_k, Y_o, Y_h,
- rad/Nm – in case of flexibility Y_b, Y_l, Y_p, Y_d.

In order to determine the stiffness matrix of the n – link systems, the system under consideration can be treated as an assembly of a two-link system with another link, etc [12]. By knowing that the forces acting at the end of the first link, in the place where the subsequent subsystems are connected (21), are also the forces acting at the beginning of the second link, and the forces acting at the end of the second link are the forces acting at the beginning of the third link, etc., therefore:

$$ \begin{cases} {}_2s_3^{(1)} + {}_2s_1^{(2)} = {}_2s_3, \\ {}_2s_4^{(1)} + {}_2s_2^{(2)} = {}_2s_4, \\ {}_2s_5^{(2)} + {}_2s_1^{(3)} = {}_2s_5, \\ {}_2s_6^{(2)} + {}_2s_2^{(3)} = {}_2s_6, \\ \ldots \\ {}_2s_{(n+3)}^{(n-1)} + {}_2s_1^{(n)} = {}_2s_{(n+3)}, \\ {}_2s_{(n+4)}^{(n-1)} + {}_2s_2^{(n)} = {}_2s_{(n+4)}. \end{cases} \tag{21} $$

Nevertheless, the principle of the inseparability of displacements provides that:

$$ \begin{cases} {}_1s_3^{(1)} \equiv {}_1s_1^{(2)} = {}_1s_3, \\ {}_1s_4^{(1)} \equiv {}_1s_2^{(2)} = {}_1s_4, \\ {}_1s_5^{(2)} \equiv {}_1s_1^{(3)} = {}_1s_5, \\ {}_1s_6^{(2)} \equiv {}_1s_2^{(3)} = {}_1s_6, \\ \quad\vdots \\ {}_1s_{(n+3)}^{(n-1)} \equiv {}_1s_1^{(n)} = {}_1s_{(n+3)}, \\ {}_1s_{(n+4)}^{(n-1)} \equiv {}_1s_2^{(n)} = {}_1s_{(n+4)}. \end{cases} \tag{22} $$

The stiffness matrix of the n – link system is determined by summing the elements of the ***Z*** stiffness matrix corresponding to the individual links of the system, by means of coordinates, connected in the places of the abrupt changes of section. In case of a system consisting of multiple links, with sectionally constant cross-section, and performing flexural vibrations, these are the four coordinates resulting from the physical properties of the system: the values of linear displacements, angular displacements, linear forces and bending moments.

The relationship (23) presents the construction of the stiffness matrix representing the n-link system.

$$ Z = \begin{bmatrix} Z_1 & & & & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_2 & & & & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & Z_n & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & \end{bmatrix} \tag{23} $$

where:

The Z_1 matrix of the first subsystem is expressed as follows:

$$ Z_1 = \begin{bmatrix} z_a^{(1)} & z_e^{(1)} & z_s^{(1)} & z_t^{(1)} \\ z_f^{(1)} & z_b^{(1)} & z_k^{(1)} & z_l^{(1)} \\ z_m^{(1)} & z_n^{(1)} & z_c^{(1)} & z_g^{(1)} \\ z_o^{(1)} & z_p^{(1)} & z_h^{(1)} & z_d^{(1)} \end{bmatrix}, $$

The Z_2 matrix of the first subsystem is expressed as follows:

$$Z_2 = \begin{bmatrix} z_a^{(2)} & z_e^{(2)} & z_s^{(2)} & z_t^{(2)} \\ z_f^{(2)} & z_b^{(2)} & z_k^{(2)} & z_l^{(2)} \\ z_m^{(2)} & z_n^{(2)} & z_c^{(2)} & z_g^{(2)} \\ z_o^{(2)} & z_p^{(2)} & z_h^{(2)} & z_d^{(2)} \end{bmatrix},$$

The Z_n matrix of the first subsystem is expressed as follows:

$$Z_n = \begin{bmatrix} z_a^{(l)} & z_e^{(l)} & z_s^{(l)} & z_t^{(l)} \\ z_f^{(l)} & z_b^{(l)} & z_k^{(l)} & z_l^{(l)} \\ z_m^{(l)} & z_n^{(l)} & z_c^{(l)} & z_g^{(l)} \\ z_o^{(l)} & z_p^{(l)} & z_h^{(l)} & z_d^{(l)} \end{bmatrix}.$$

The final result of the (23) relationship is the $\boldsymbol{Z}$ stiffness matrix dimensioned n×s, where: n – the number of subsystems of the continuous system, s – the number of coordinates of the generalised system. Various boundary conditions imposed on the ends of links need to be taken into account to determine the dynamic characteristics of real mechanical systems. The dynamic stiffness matrices of the system do not have to be modified when changing the boundary conditions. It is sufficient to provide the relevant flexibilities of the system for the stiffness matrix of the system.

3. Systems with the discrete-continuous distribution of parameters

A mechanical system needs to be discomposed to two subsystems to present a vibration mechanical system with the discrete-continuous distribution of parameters as a hybrid graph [14, 15], i.e. a subsystem with aggregated parameters modelling with hybrid graphs and a subsystem with the continuous distribution of parameters modelled with flow graphs and then reduced to the edge of a hybrid graph. Dynamic flexibility corresponding to the edge of coincidence with a discrete system is assigned to this edge, whereas a continuous subsystem may contain elements vibrating longitudinally, torsionally, flexurally, longitudinally–flexurally or longitudinally–flexurally–torsionally. The idea of analysing the systems with the discrete-continuous distribution of parameters with the non-classical method is shown in Fig. 5. Note that each of the subsystems of a complex mechanical system can be considered separately.

A phenomenological model of a discrete-continuous mechanical system was adopted without influencing the generality of the considerations. The following was distinguished between for a physical model: l elements with the parameters distributed continuously and with sectionally constant cross-section connected in the n_2 coincidence points with a subsystem with the discrete distribution of parameters, n_1 inertial elements, n_3 kinematic excitations, n_4 elastic and dampening elements (of the c and b type) and the n_5 force excitations. A general form of a matrix flow graph was obtained by using the theory of hybrid graphs and matrix hybrid graphs [10, 13, 14].

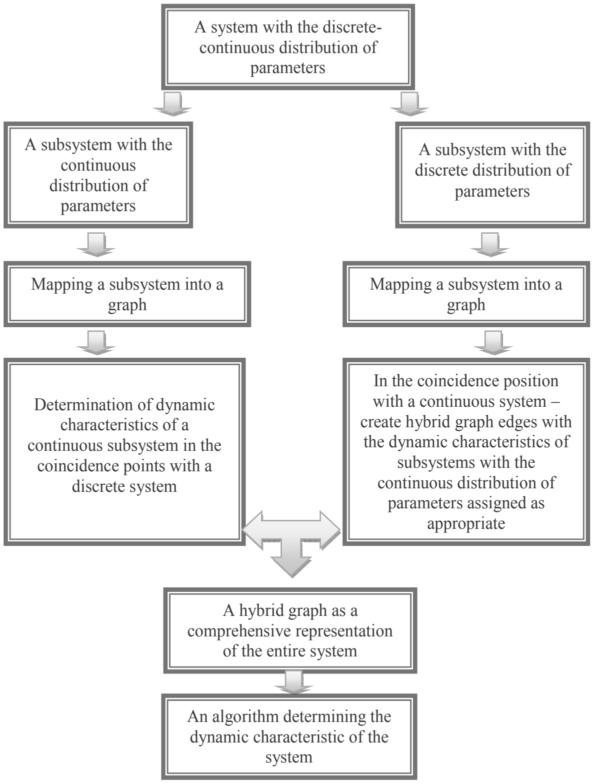

Fig. 5. An idea of an analysis of systems with the discrete-continuous distribution of parameters with the hybrid graphs method

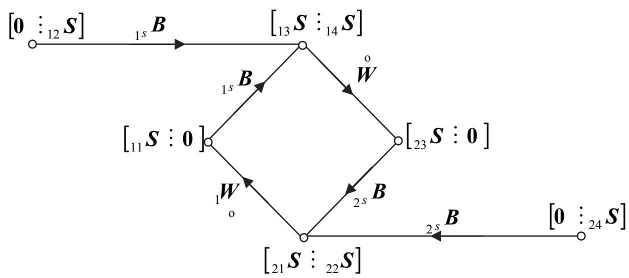

Fig. 6. A general form of a matrix flow graph

The meanings of the symbol in Fig. 6 are: ${}_{11}S$ – the matrix of the polar coordinates of the passive branches of the graph, ${}_{12}S$ – the matrix of the generalized polar coordinates of the active branches of the graph (generated by the kinematical excitations), ${}_{13}S$ – the matrix of linear and angular displacements of the elements of type of c and b, ${}_{14}S$ – the matrix of linear and angular displacements of the active branches of the graph (generated by the dynamic excitations), ${}_{21}S$ – the matrix of the flow coordinates of the passive branches of the graph, ${}_{22}S$ – the matrix of the flow coordinates of the active branches of the graph (generated by the kinematical excitations), ${}_{23}S$ – the matrix of the flow coordinates of the passive principal branches of the graph, ${}_{24}S$ – the matrix of the flow coordinates of the active branches of the graph (generated by the dynamic excitations), ${}_{1s}\tilde{\boldsymbol{B}}$ – polar variable distribution matrix, ${}_{2s}\tilde{\boldsymbol{B}}$ – flow variables matrix, ${}_{1}\underset{o}{\boldsymbol{W}}$ – tree elements dynamic flexibility matrix, $\overset{o}{\boldsymbol{W}}$ – co-tree elements dynamic ridigidy matrix.

4 formulas (transition functions) can be distinguished between based on the flow graph obtained that are expressing the matrix-based dynamic characteristics of the studied vibration mechanical system with the discrete-continuous distribution of parameters. The characteristics are described with the following relationships:

- The operational, matrix-based, dynamic characteristic of force excitations' transformation into the variable flow branches of a hybrid graph.

$$\boldsymbol{Y}_1 = {}_{2s}\boldsymbol{B}\left[\mathbf{1} - {}_{1}\underset{o}{\boldsymbol{W}}\,{}_{1s}\boldsymbol{B}\overset{o}{\boldsymbol{W}}\,{}_{2s}\boldsymbol{B}\right]^{-1} \tag{24}$$

- The operational, matrix-based, dynamic characteristic of force excitations' transformation into the variable polar branches of a hybrid graph.

$$\boldsymbol{Y}_2 = {}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\left[\mathbf{1} - {}_{1s}\boldsymbol{B}\overset{o}{\boldsymbol{W}}\,{}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\right]^{-1} \tag{25}$$

- The operational, matrix-based, dynamic characteristic of force excitations' transformation into the variable flow principal branches of a hybrid graph.

$$\boldsymbol{Y}_3 = {}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\,{}_{1s}\boldsymbol{B}\left[\mathbf{1} - \overset{o}{\boldsymbol{W}}\,{}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\,{}_{1s}\boldsymbol{B}\right]^{-1} \tag{26}$$

- The operational, matrix-based, dynamic characteristic of force excitations' transformation into the variable polar principal branches of a hybrid graph.

$$\boldsymbol{Y}_4 = {}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\,{}_{1s}\boldsymbol{B}\overset{o}{\boldsymbol{W}}\left[\mathbf{1} - \overset{o}{\boldsymbol{W}}\,{}_{2s}\boldsymbol{B}\,{}_{1}\underset{o}{\boldsymbol{W}}\,{}_{1s}\boldsymbol{B}\overset{o}{\boldsymbol{W}}\right]^{-1} \tag{27}$$

Accordingly, on the bases of the information on the dynamic characteristics of the system for any inputs or outputs in the form of kinematical and dynamic excitations it is possible to designate the frequency characteristics for the discussed class of systems.

4. The system of the research

The system considered is made up of two rod elements with sectionally constant cross-section, performing flexural vibrations and of two inertial elements excited with kinematic excitations.

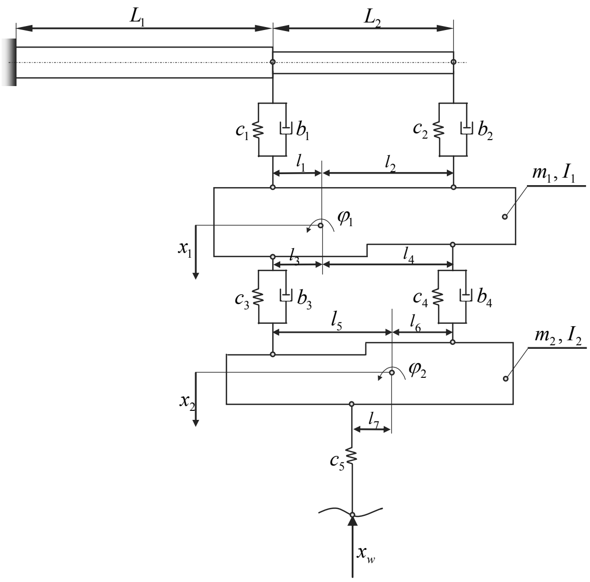

Fig. 7. The analysed system with the characteristic values provided

The parameters provided in Table 1 (parameters of a subsystem with the continuous distribution of parameters), 2 (inertial parameters), 3 (elasticity and dampening parameters) and 4 and 5 (geometric parameters) were adopted for the model presented in the Figure 7.

Table 1.
Parameters of a subsystem with the continuous distribution of parameters

$L\,[m]$		$d\,[m]$		material
L_1	L_2	d_1	d_2	
0.5	0.75	0.05	0.03	steel

Table 2.
Inertial parameters

$m\,[kg]$		$I\,[kg \cdot m^2]$	
m_1	m_2	I_1	I_2
10	14	1.0416	1.365

Table 3.
Elasticity and dampening parameters

$c\left[\frac{N}{m}\right]$				
c_1	c_2	c_3	c_4	c_5
$1\cdot10^6$	$1.1\cdot10^6$	$1.2\cdot10^6$	$1.1\cdot10^6$	$1\cdot10^6$

$b\left[\frac{N\cdot s}{m}\right]$			
b_1	b_2	b_3	b_4
$0.18\cdot10^4$	$0.2\cdot10^4$	$0.17\cdot10^4$	$0.15\cdot10^4$

Table 4.
Geometric parameters

the distance of coincidence points EST* from the centre of mass of an inertia element 1.			
l_1	l_2	l_3	l_4
0.3	0.4	0.3	0.4

*EST – an elastic-dampening element.

Table 5.
Geometric parameters

the distance of coincidence points EST* from the centre of mass of an inertia element 2.		
l_5	l_6	l_7
0.4	0.45	0.2

*EST – an elastic-dampening element.

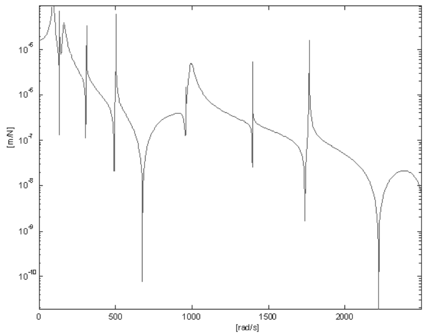

Fig. 8. A diagram of the dynamic flexibility of a subsystem with the continuous distribution of parameters in the first point of coincidence with the subsystem with the discrete distribution of parameters

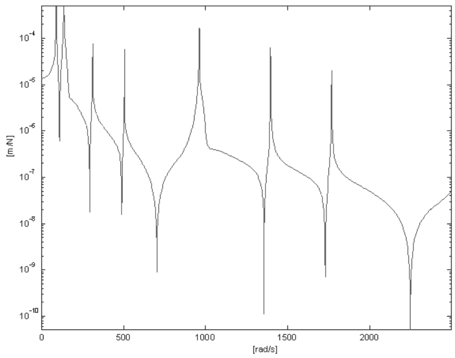

Fig. 9. A diagram of the dynamic flexibility of a subsystem with the continuous distribution of parameters in the second point of coincidence with the subsystem with the discrete distribution of parameters

A diagram of the dynamic flexibility of a subsystem with the continuous distribution of parameters in the first and second point of coincidence with the subsystem with the discrete distribution of parameters is shown in Figures 8 and 9, while Figure 10 shown diagram of dynamic flexibility between a force acting in the second point of coincidence of subsystems and the displacement in the first point of coincidence.

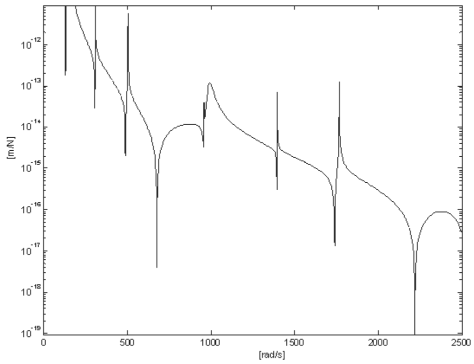

Fig. 10. A diagram of dynamic flexibility between a force acting in the second point of coincidence of subsystems and the displacement in the first point of coincidence

5. Conclusions

By using the non-classical methods of modelling, it was possible to develop a method of determining the matrices (flexibilities) of multi-link vibration mechanical systems with the continuous distribution of parameters that are able to perform longitudinal and flexural vibrations. This has shortened and simplified the computing process markedly. The modelling and analysis of discrete–continuous vibration systems with conjugations using the non-classical method is a more general approach as compared to modelling and analysis in classical terms. A simple task of dynamics can be solved by using this method, without limitations deriving from the type and number of the elements of a mechanical system. The considerations presented in the paper are obviously not exhausting all the issues associated with an analysis of the considered class of systems, therefore, other, open issues for future research arise, as well, and the considerations may either generalise or represent an assumption for undertaking new considerations. The issues may concern: extending the considered class of systems with spatial systems; introducing additional kinematic excitations in form of a function of speed and accelerations; enabling the generation of time responses to any excitations; extending the method presented to cover the investigation of non-linear systems.

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